MathTrip

La pente, déjà verticale, se redressait encore.
The slope, already vertical, was still rising.

Georges Livanos (french alpinist)

Few formulæ and mathematical facts

For fun and to show the font

" GFS NeoHellenic"

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Mathematical formulæ and facts

Definitions

$f(n) = O(g(n))$	iff $\exists$ positive c, n_0 such that $0 \le f(n) \le cg(n)$ $\forall n \ge n_0$.
$f(n) = \Omega(g(n))$	iff $\exists$ positive c, n_0 such that $f(n) \ge cg(n) \ge 0$ $\forall n \ge n_0$.
$f(n) = \Theta(g(n))$	iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.
$f(n) = o(g(n))$	iff $\lim_{n\to\infty} f(n)/g(n) = 0$.
$\lim_{n\to\infty} a_n = a$	iff $\forall \epsilon > 0$, $\exists n_0$ such that $\lvert a_n - a\rvert < \epsilon$, $\forall n \ge n_0$.
$\sup S$	least $b \in \mathbb{R}$ such that $b \ge s$, $\forall s \in S$.
$\inf S$	greatest $b \in \mathbb{R}$ such that $b \le s$, $\forall s \in S$.
$\liminf_{n\to\infty} a_n$	$\lim_{n\to\infty} \inf\{a_i \mid i \ge n, i \in \mathbb{N}\}$
$\limsup_{n\to\infty} a_n$	$\lim_{n\to\infty} \sup\{a_i \mid i \ge n, i \in \mathbb{N}\}$
$\binom{n}{k}$	Combinations: Size k subsets of a size n set.
$\left[{n \atop k}\right]$	Stirling numbers (1st kind): Arrangements of an n element set into k cycles.
$\left\{{n \atop k}\right\}$	Stirling numbers (2nd kind): Partitions of an n element set into k non-empty sets.
$\left\langle{n \atop k}\right\rangle$	1st order Eulerian numbers: Permutations $\pi_1\pi_2\ldots\pi_n$ on $\{1, 2, \ldots, n\}$ with k ascents.
$\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle$	2nd order Eulerian numbers.
C_n	Catalan Numbers: Binary trees with $n+1$ vertices.

Series

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

In general:

$$\sum_{i=1}^{n} i^m = \frac{1}{m+1}\left[(n+1)^{m+1} - 1 - \sum_{i=1}^{n}\Big((i+1)^{m+1} - i^{m+1} - (m+1)i^m\Big)\right]$$

$$\sum_{i=1}^{n-1} i^m = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k n^{m+1-k}$$

Geometric series:

$$\sum_{i=0}^{n} c^i = \frac{1-c^{n+1}}{1-c}, \quad c \ne 1, \quad \sum_{i=0}^{\infty} c^i = \frac{1}{1-c}, \quad \sum_{i=1}^{\infty} c^i = \frac{c}{1-c}, \quad |c| < 1,$$

$$\sum_{i=0}^{n} ic^i = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}, \quad c \ne 1, \quad \sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}, \quad |c| < 1$$

Harmonic series:

$$H_n = \sum_{i=1}^{n}\frac{1}{i}, \quad \sum_{i=1}^{n} iH_i = \frac{n(n+1)}{2}H_n - \frac{n(n-1)}{4},$$

$$\sum_{i=1}^{n} H_i = (n+1)H_n - n, \quad \sum_{i=1}^{n}\binom{i}{m}H_i = \binom{n+1}{m+1}\left(H_{n+1} - \frac{1}{m+1}\right)$$

Identities

1. $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ **2.** $\sum_{k=0}^{n}\binom{n}{k} = 2^n$ **3.** $\binom{n}{k} = \binom{n}{n-k}$

4. $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$ **5.** $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

6. $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ **7.** $\sum_{k=0}^{n}\binom{r+k}{k} = \binom{r+n+1}{n}$

8. $\sum_{k=0}^{n}\binom{k}{m} = \binom{n+1}{m+1}$ **9.** $\sum_{k=0}^{n}\binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$

10. $\binom{n}{k} = (-1)^k\binom{k-n-1}{k}$ **11.** $\left\{{n \atop 1}\right\} = \left\{{n \atop n}\right\} = 1$ **12.** $\left\{{n \atop 2}\right\} = 2^{n-1} - 1$

13. $\left\{{n \atop k}\right\} = k\left\{{n-1 \atop k}\right\} + \left\{{n-1 \atop k-1}\right\}$ **14.** $\left[{n \atop 1}\right] = (n-1)!$ **15.** $\left[{n \atop 2}\right] = (n-1)!H_{n-1}$

16. $\left[{n \atop n}\right] = 1$ **17.** $\left[{n \atop k}\right] \ge \left\{{n \atop k}\right\}$ **18.** $\left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right]$

19. $\left\{{n \atop n-1}\right\} = \left[{n \atop n-1}\right] = \binom{n}{2}$ **20.** $\sum_{k=0}^{n}\left[{n \atop k}\right] = n!$ **21.** $C_n = \frac{1}{n+1}\binom{2n}{n}$ **22.** $\left\langle{n \atop 0}\right\rangle = \left\langle{n \atop n-1}\right\rangle = 1$

23. $\left\langle{n \atop k}\right\rangle = \left\langle{n \atop n-1-k}\right\rangle$ **24.** $\left\langle{n \atop k}\right\rangle = (k+1)\left\langle{n-1 \atop k}\right\rangle + (n-k)\left\langle{n-1 \atop k-1}\right\rangle$ **25.** $\left\langle{0 \atop k}\right\rangle = \begin{cases} 1 & \text{if k=0,} \\ 0 & \text{otherwise} \end{cases}$

26. $\left\langle{n \atop 1}\right\rangle = 2^n - n - 1$ **27.** $\left\langle{n \atop 2}\right\rangle = 3^n - (n+1)2^n + \binom{n+1}{2}$ **28.** $x^n = \sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\binom{x+k}{n}$

29. $\left\langle{n \atop m}\right\rangle = \sum_{k=0}^{m}\binom{n+1}{k}(m+1-k)^n(-1)^k$ **30.** $m!\left\{{n \atop m}\right\} = \sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\binom{k}{n-m}$ **31.** $\left\langle{n \atop m}\right\rangle = \sum_{k=0}^{n}\left\{{n \atop k}\right\}\binom{n-k}{m}(-1)^{n-k-m}k!$

32. $\left\langle\!\left\langle{n \atop 0}\right\rangle\!\right\rangle = 1$ **33.** $\left\langle\!\left\langle{n \atop n}\right\rangle\!\right\rangle = 0$, for $n \ne 0$ **34.** $\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle = (k+1)\left\langle\!\left\langle{n-1 \atop k}\right\rangle\!\right\rangle + (2n-1-k)\left\langle\!\left\langle{n-1 \atop k-1}\right\rangle\!\right\rangle$ **35.** $\sum_{k=0}^{n}\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle = \frac{(2n)^{\underline{n}}}{2^n}$

36. $\left\{{x \atop x-n}\right\} = \sum_{k=0}^{n}\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle\binom{x+n-1-k}{2n}$ **37.** $\left\{{n+1 \atop m+1}\right\} = \sum_{k}\binom{n}{k}\left\{{k \atop m}\right\} = \sum_{k=0}^{n}\left\{{k \atop m}\right\}(m+1)^{n-k}$

Mathematical formulæ and facts

Identities Cont.

38. $\begin{bmatrix} n+1 \\ m+1 \end{bmatrix} = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} \binom{k}{m} = \sum_{k=0}^{n} \begin{bmatrix} k \\ m \end{bmatrix} n^{\underline{n-k}} = n! \sum_{k=0}^{n} \frac{1}{k!} \begin{bmatrix} k \\ m \end{bmatrix}$ **39.** $\begin{bmatrix} x \\ x-n \end{bmatrix} = \sum_{k=0}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle \binom{x+k}{2n}$

40. $\begin{Bmatrix} n \\ m \end{Bmatrix} = \sum_k \binom{n}{k} \begin{Bmatrix} k+1 \\ m+1 \end{Bmatrix} (-1)^{n-k}$ **41.** $\begin{bmatrix} n \\ m \end{bmatrix} = \sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \binom{k}{m} (-1)^{m-k}$ **42.** $\begin{Bmatrix} m+n+1 \\ m \end{Bmatrix} = \sum_{k=0}^{m} k \begin{Bmatrix} n+k \\ k \end{Bmatrix}$

43. $\begin{bmatrix} m+n+1 \\ m \end{bmatrix} = \sum_{k=0}^{m} k(n+k) \begin{bmatrix} n+k \\ k \end{bmatrix}$ **44.** $\binom{n}{m} = \sum_k \begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix} \begin{bmatrix} k \\ m \end{bmatrix} (-1)^{m-k}$

45. $(n-m)! \binom{n}{m} = \sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \begin{Bmatrix} k \\ m \end{Bmatrix} (-1)^{m-k}$, for $n \geq m$ **46.** $\begin{Bmatrix} n \\ n-m \end{Bmatrix} = \sum_k \binom{m-n}{m+k} \binom{m+n}{n+k} \begin{bmatrix} m+k \\ k \end{bmatrix}$

47. $\begin{bmatrix} n \\ n-m \end{bmatrix} = \sum_k \binom{m-n}{m+k} \binom{m+n}{n+k} \begin{Bmatrix} m+k \\ k \end{Bmatrix}$ **48.** $\begin{Bmatrix} n \\ \ell+m \end{Bmatrix} \binom{\ell+m}{\ell} = \sum_k \begin{Bmatrix} k \\ \ell \end{Bmatrix} \begin{Bmatrix} n-k \\ m \end{Bmatrix} \binom{n}{k}$

49. $\begin{bmatrix} n \\ \ell+m \end{bmatrix} \binom{\ell+m}{\ell} = \sum_k \begin{bmatrix} k \\ \ell \end{bmatrix} \begin{bmatrix} n-k \\ m \end{bmatrix} \binom{n}{k}$

Trees

Every tree with n vertices has $n-1$ edges.

Kraft inequality:

If the depths of the leaves of a binary tree are $d_1 \cdots d_n$: $\sum_{i=1}^{n} 2^{-d_i} \leq 1$, and equality holds only if every internal node has 2 sons.

Recurrences

Master method:

$T(n) = aT(n/b) + f(n)$ $a \geq 1, b > 1$

If $\exists \epsilon > 0$ such that $f(n) = O(n^{\log_b a - \epsilon})$ then:

$T(n) = \Theta(n^{\log_b a})$

If $f(n) = \Theta(n^{\log_b a})$ then

$T(n) = \Theta(n^{\log_b a} \log_2 n)$

If $\exists \epsilon > 0$ such that $f(n) = \Omega(n^{\log_b a + \epsilon})$, and $\exists c < 1$ such that $af(n/b) \leq cf(n)$ for large n, then:

$T(n) = \Theta(f(n))$

Substitution (example):

Consider the following recurrence:

$T_{i+1} = 2^{2^i} \cdot T_i^2$, $T_1 = 2$. Note that T_i is always a power of two.

Let $t_i = \log_2 T_i$. Then we have:

$t_{i+1} = 2^i + 2t_i$, $t_1 = 1$

Let $u_i = t_i / 2^i$. Dividing both sides of the previous equation by 2^{i+1} we get:

$\frac{t_{i+1}}{2^{i+1}} = \frac{2^i}{2^{i+1}} + \frac{t_i}{2^i}$

Substituting we find:

$u_{i+1} = 2^{-1} + u_i$, $u_1 = 2^{-1}$, which is simply $u_i = i/2$.

So we find that T_i has the closed form $T_i = 2^{i2^{i-1}}$.

Summing factors (example):

Consider the following recurrence:

$T(n) = 3T(n/2) + n$, $T(1) = 1$

Rewrite so that all terms involving T are on the left side:

$T(n) - 3T(n/2) = n$

Now expand the recurrence, and choose a factor which makes the left side "telescope".

$$\begin{aligned}
(T(n) - 3T(n/2) &= n) \\
(T(n/2) - 3T(n/4) &= n/2) \\
&\vdots \\
3^{\log_2 n - 1}(T(2) - 3T(1) &= 2)
\end{aligned}$$

Let $m = \log_2 n$. Summing the left side we get:

$$\begin{aligned}
T(n) - 3^m T(1) &= T(n) - 3^m \\
&= T(n) - n^k
\end{aligned}$$

where $k = \log_2 3 \approx 1{,}58496$.

Summing the right side we get:

$$\sum_{i=0}^{m-1} \frac{n}{2^i} 3^i = n \sum_{i=0}^{m-1} \left(\frac{3}{2}\right)^i$$

Let $c = \frac{3}{2}$. Then we have:

$$\begin{aligned}
n \sum_{i=0}^{m-1} c^i &= n \left(\frac{c^m - 1}{c - 1} \right) \\
&= 2n \left(c^{\log_2 n} - 1 \right) \\
&= 2n \left(c^{(k-1)\log_c n} - 1 \right) \\
&= 2n^k - 2n \text{ and so}
\end{aligned}$$

$T(n) = 3n^k - 2n$.

Full history recurrences can often be changed to limited history ones.

Example:

Consider: $T_i = 1 + \sum_{j=0}^{i-1} T_j$, $T_0 = 1$

Note that:

$$T_{i+1} = 1 + \sum_{j=0}^{i} T_j$$

By subtracting we find:

$$\begin{aligned}
T_{i+1} - T_i &= 1 + \sum_{j=0}^{i} T_j - 1 - \sum_{j=0}^{i-1} T_j \\
&= T_i
\end{aligned}$$

And so $T_{i+1} = 2T_i = 2^{i+1}$.

Generating functions:

1. Multiply both sides of the equation by x^i.
2. Sum both sides over all i for which the equation is valid.
3. Choose a generating function $G(x)$. Usually $G(x) = \sum_{i=0}^{\infty} x^i g_i$.
4. Rewrite the equation in terms of the generating function $G(x)$.
5. Solve for $G(x)$.
6. The coefficient of x^i in $G(x)$ is g_i.

Example:

Let the equation:

$g_{i+1} = 2g_i + 1$, $g_0 = 0$.

Multiply and sum:

$\sum_{i \geq 0} g_{i+1} x^i = \sum_{i \geq 0} 2g_i x^i + \sum_{i \geq 0} x^i$ We choose: $G(x) = \sum_{i \geq 0} x^i g_i$.

Rewrite in terms of $G(x)$:

$$\frac{G(x) - g_0}{x} = 2G(x) + \sum_{i \geq 0} x^i$$

Simplify:

$$\frac{G(x)}{x} = 2G(x) + \frac{1}{1-x}$$

Solve for $G(x)$:

$$G(x) = \frac{x}{(1-x)(1-2x)}.$$

Expand this using partial fractions:

$$\begin{aligned}
G(x) &= x \left(\frac{2}{1-2x} - \frac{1}{1-x} \right) \\
&= x \left(2 \sum_{i \geq 0} 2^i x^i - \sum_{i \geq 0} x^i \right) \\
&= \sum_{i \geq 0} (2^{i+1} - 1) x^{i+1}
\end{aligned}$$

So $g_i = 2^i - 1$.

Mathematical formulæ and facts

$\pi \approx 3{,}14159$ $e \approx 2{,}71828$ $\gamma \approx 0{,}57721$ $\phi = \frac{1+\sqrt{5}}{2} \approx 1{,}61803$ $\hat{\phi} = \frac{1-\sqrt{5}}{2} \approx -0{,}61803$

i	2^i	p_i
1	2	2
2	4	3
3	8	5
4	16	7
5	32	11
6	64	13
7	128	17
8	256	19
9	512	23
10	1 024	29
11	2 048	31
12	4 096	37
13	8 192	41
14	16 384	43
15	32 768	47
16	65 536	53
17	131 072	59
18	262 144	61
19	524 288	67
20	1 048 576	71
21	2 097 152	73
22	4 194 304	79
23	8 388 608	83
24	16 777 216	89
25	33 554 432	97
26	67 108 864	101
27	134 217 728	103
28	268 435 456	107
29	536 870 912	109
30	1 073 741 824	113
31	2 147 483 648	127
32	4 294 967 296	131

General

Bernoulli Numbers ($B_i = 0$,odd$i \neq 1$):

$B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$,

$B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = -\frac{1}{30}$,

$B_{10} = \frac{5}{66}$

Change of base, quadratic formula:

$$\log_b x = \frac{\log_a x}{\log_a b}, \quad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Euler's number e:

$$e = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!},$$

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x,$$

$$\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1},$$

$$\left(1 + \frac{1}{n}\right)^n = e - \frac{e}{2n} + \frac{11e}{24n^2} - O\left(\frac{1}{n^3}\right)$$

Harmonic numbers:

$$1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \frac{49}{20}, \frac{363}{140}, \frac{761}{280}, \frac{7129}{2520}, \ldots, \ln n < H_n < \ln n + 1,$$

$$H_n = \ln n + \gamma + O\left(\frac{1}{n}\right)$$

Factorial, Stirling's approximation:

1, 2, 6, 24, 120, 720, 5040, 40320, 362880, ...

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right)$$

Ackermann's function and inverse:

$$a(i,j) = \begin{cases} 2^j & i = 1 \\ a(i-1, 2) & j = 1 \\ a(i-1, a(i, j-1)) & i, j \geq 2 \end{cases}$$

$$\alpha(i) = \min\{j \mid a(j,j) \geq i\}$$

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

Probability

Binomial distribution:

$$\Pr[X = k] = \binom{n}{k} p^k q^{n-k}$$

$$q = 1 - p,$$

$$\mathbb{E}[X] = \sum_{k=1}^{n} k \binom{n}{k} p^k q^{n-k} = np$$

Poisson distribution:

$$\Pr[X = k] = \frac{e^{-\lambda}\lambda^k}{k!}, \ \mathbb{E}[X] = \lambda$$

Probability cont.

Normal (Gaussian) distribution:

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \ \mathbb{E}[X] = \mu$$

Continuous distributions:

If $\Pr[a < X < b] = \int_a^b p(x)\,dx$, then p is the probability density function of X.
If $\Pr[X < a] = P(a)$, then P is the distribution function of X.
If P and p both exist then $P(a) = \int_{-\infty}^{a} p(x)\,dx$.

Expectation:

If X is discrete $\mathbb{E}[g(X)] = \sum_x g(x) \Pr[X = x]$.
If X continuous then

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)p(x)\,dx = \int_{-\infty}^{\infty} g(x)\,dP(x).$$

Variance, standard deviation:

$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$, $\sigma = \sqrt{\mathrm{Var}[X]}$

For events A and B:

$\Pr[A \vee B] = \Pr[A] + \Pr[B] - \Pr[A \ \& \ B]$

iff A and B are independent:

$$\Pr[A \ \& \ B] = \Pr[A] \cdot \Pr[B]$$

$$\Pr[A|B] = \frac{\Pr[A \ \& \ B]}{\Pr[B]}$$

For random variables X and Y:

if X and Y are independent:

$$\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$$

$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$ $\mathbb{E}[cX] = c\mathbb{E}[X]$

Bayes' theorem:

$$\Pr[A_i|B] = \frac{\Pr[B|A_i]\Pr[A_i]}{\sum_{j=1}^{n} \Pr[A_j]\Pr[B|A_j]}$$

Inclusion-exclusion:

$$\Pr\left[\bigvee_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \Pr[X_i] + \sum_{k=2}^{n} (-1)^{k+1} \sum_{i_i < \cdots < i_k} \Pr\left[\bigwedge_{j=1}^{k} X_{i_j}\right]$$

Moment inequalities:

$$\Pr\left[|X| \geq \lambda \mathbb{E}[X]\right] \leq \frac{1}{\lambda},$$

$$\Pr\left[\left|X - \mathbb{E}[X]\right| \geq \lambda \cdot \sigma\right] \leq \frac{1}{\lambda^2}$$

Geometric distribution:

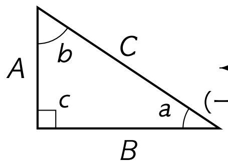
$\Pr[X = k] = pq^{k-1}$, $q = 1 - p$,

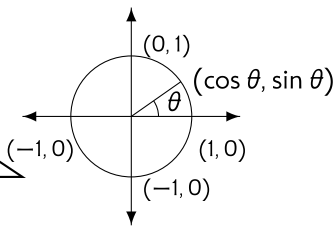
$$\mathbb{E}[X] = \sum_{k=1}^{\infty} kpq^{k-1} = \frac{1}{p}$$

The "coupon collector": We are given a random coupon each day, and there are n different types of coupons. The distribution of coupons is uniform. The expected number of days to pass before we to collect all n types is $n = H_n$.

Mathematical formulæ and facts

Trigonometry





Pythagorean theorem:
$C^2 = A^2 + B^2$.

Definitions:

$\sin a = \frac{A}{C}$ $\quad \cos a = \frac{B}{C}$

$\csc a = \frac{C}{A}$ $\quad \sec a = \frac{C}{B}$

$\tan a = \frac{\sin a}{\cos a} = \frac{A}{B}$ $\quad \cot a = \frac{\cos a}{\sin a} = \frac{B}{A}$

Area, radius of inscribed circle:
$\frac{1}{2}AB \quad \frac{AB}{A+B+C}$

Identities:

$\sin x = \frac{1}{\csc x}$, $\cos x = \frac{1}{\sec x}$, $\tan x = \frac{1}{\cot x}$,
$\sin^2 x + \cos^2 x = 1$, $1 + \tan^2 x = \sec^2 x$,
$1 + \cot^2 x = \csc^2 x$, $\sin x = \cos\left(\frac{\pi}{2} - x\right)$,
$\sin x = \sin(\pi - x)$, $\cos x = -\cos(\pi - x)$,
$\tan x = \cot\left(\frac{\pi}{2} - x\right)$,
$\cot x = -\cot(\pi - x)$,
$\csc x = \cot\frac{x}{2} - \cot x$,
$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$,
$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$,
$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$,
$\sin 2x = 2 \sin x \cos x$, $\sin 2x = \frac{2 \tan x}{1 + \tan^2 x}$,
$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot x \pm \cot y}$,
$\cos 2x = \cos^2 x - \sin^2 x$,
$\cos 2x = 2\cos^2 x - 1$,
$\cos 2x = 1 - 2\sin^2 x$, $\cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$,
$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$, $\cot 2x = \frac{\cot^2 x - 1}{2 \cot x}$,
$\sin(x + y)\sin(x - y) = \sin^2 x - \sin^2 y$,
$\cos(x + y)\cos(x - y) = \cos^2 x - \sin^2 y$.

Euler's equation:
$e^{ix} = \cos x + i \sin x$, $e^{i\pi} + 1 = 0$.

Matrices

Multiplication:
$C = A \cdot B$, $c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}$.

Determinants:
$\det A \neq 0$ iff A is non-singular.
$\det A \cdot B = \det A \cdot \det B$,
$\det A = \sum_{\pi} \prod_{i=1}^{n} \mathrm{sign}(\pi) a_{i,\pi(i)}$.

2×2 and 3×3 determinant:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = g \begin{bmatrix} b & c \\ e & f \end{bmatrix} - h \begin{bmatrix} a & c \\ d & f \end{bmatrix} + i \begin{bmatrix} a & b \\ d & e \end{bmatrix}$$

$$\det A = aei + bfg + cdh - ceg - fha - ibd$$

Permanents:
$\mathrm{perm}\, A = \sum_{\pi} \prod_{i=1}^{n} a_{i,\pi(i)}$.

Hyperbolic Functions

Definitions:

$\sinh x = \frac{e^x - e^{-x}}{2}$ $\quad \cosh x = \frac{e^x + e^{-x}}{2}$

$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ $\quad \mathrm{csch}\, x = \frac{1}{\sinh x}$

$\mathrm{sech}\, x = \frac{1}{\cosh x}$ $\quad \coth x = \frac{1}{\tanh x}$

Identities:

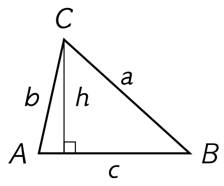
$\cosh^2 x - \sinh^2 x = 1$,
$\tanh^2 x + \mathrm{sech}^2 x = 1$,
$\coth^2 x - \mathrm{csch}^2 x = 1$, $\sinh(-x) = -\sinh x$,
$\cosh(-x) = \cosh x$, $\tanh(-x) = -\tanh x$,
$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$,
$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$,
$\sinh 2x = 2 \sinh x \cosh x$,
$\cosh 2x = \cosh^2 x + \sinh^2 x$,
$\cosh x + \sinh x = e^x$, $\cosh x - \sinh x = e^{-x}$,
$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$, $n \in \mathbb{Z}$,
$2 \sinh^2 \frac{x}{2} = \cosh x - 1$,
$2 \cosh^2 \frac{x}{2} = \cosh x + 1$.

$\sin 0 = 0, \sin\frac{\pi}{6} = \frac{1}{2}, \quad \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}, \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}, \sin\frac{\pi}{2} = 1$

$\cos 0 = 1, \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}, \cos\frac{\pi}{3} = \frac{1}{2}, \quad \cos\frac{\pi}{2} = 0$

$\tan 0 = 1, \tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}, \tan\frac{\pi}{4} = 1, \quad \tan\frac{\pi}{3} = \sqrt{3}, \tan\frac{\pi}{2} = \infty$

More Trig.



Law of cosines:

$c^2 = a^2 + b^2 - 2ab \cos C$.

Area:
$A = \frac{1}{2}hc = \frac{1}{2}ab \sin C = \frac{c^2 \sin A \sin B}{2 \sin C}$

Heron's formula:
$A = \sqrt{s \cdot s_a \cdot s_b \cdot s_c}$
$s = \frac{1}{2}(a + b + c)$
$s_a = s - a$ $s_b = s - b$
$s_c = s - c$

More identities:

$\sin\frac{x}{2} = \sqrt{\frac{1 - \cos x}{2}}$,
$\cos\frac{x}{2} = \sqrt{\frac{1 + \cos x}{2}}$,
$\tan\frac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}} = \frac{1 - \cos x}{\sin x}, = \frac{\sin x}{1 + \cos x}$,
$\cot\frac{x}{2} = \sqrt{\frac{1 + \cos x}{1 - \cos x}} = \frac{1 + \cos x}{\sin x}, = \frac{\sin x}{1 - \cos x}$,
$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$,
$\cos x = \frac{e^{ix} + e^{-ix}}{2}$,
$\tan x = -i\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}} = -i\frac{e^{2ix} - 1}{e^{2ix} + 1}$,
$\sin x = \frac{\sinh ix}{i}$,
$\cos x = \cosh ix$,
$\tan x = \frac{\tanh ix}{i}$.

... in mathematics you don't understand things, you just get used to them.
— J. von Neumann

Mathematical formulæ and facts

Number Theory

The Chinese remainder theorem:
There exists a number C such that:

$$C \equiv r_1 \mod m_1$$
$$\vdots$$
$$C \equiv r_n \mod m_n$$

if m_i and m_j are relatively prime for $i \neq j$.

Euler's function:
$\phi(x)$ is the number of positive integers less than x relatively prime to x.
If $\prod_{i=1}^n p_i^{e_i}$ is the prime factorization of x then
$\phi(x) = \prod_{i=1}^n p_i^{e_i - 1}(p_i - 1)$

Euler's theorem:
If a and b are relatively prime then
$1 \equiv a^{\phi(b)} \bmod b$

Fermat's theorem:
$1 \equiv a^{p-1} \bmod p$

The Euclidean algorithm:
if $a > b$ are integers then
$gcd(a, b) = \gcd(a \bmod b, b)$

If $\prod_{i=1}^n p_i^{e_i}$ is the prime factorization of x then
$S(x) = \sum_{d|x} d = \prod_{i=1}^n \frac{p_i^{e_i+1} - 1}{p_i - 1}$

Perfect Numbers:
x is an even perfect number iff
$x = 2^{n-1}(2^n - 1)$ and $2^n - 1$ is prime.

Wilson's theorem:
n is a prime iff $(n-1)! \equiv -1 \bmod n$.

Möbius inversion:

$$\mu(i) = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{if } i \text{ is not square-free} \\ (-1)^r & \text{if } i \text{ is the product of} \\ & ir \text{ distinct primes.} \end{cases}$$

If $G(a) = \sum_{d|a} F(d)$ then
$F(a) = \sum_{d|a} \mu(d) G\left(\frac{a}{d}\right)$

Prime numbers:

$$p_n = \ln n + n \ln\ln n - n + n \frac{\ln\ln n}{\ln n} + O\left(\frac{n}{\ln n}\right)$$

$$\pi(n) = \frac{n}{\ln n} + \frac{n}{(\ln n)^2} + \frac{2!n}{(\ln n)^3} + O\left(\frac{n}{(\ln n)^4}\right)$$

Graph Theory

Definitions:

Loop	An edge connecting a vertex to itself.
Directed	Each edge has a direction.
Simple	Graph with no loops or multi-edges.
Walk	A sequence $v_0 e_1 v_1 \ldots e_\ell v_\ell$.
Trail	A walk with distinct edges.
Path	A trail with distinct vertices.
Connected	A graph where there exists a path between any two vertices.
Component	A maximal connected subgraph.
Tree	A connected acyclic graph.
Free tree	A tree with no root.
DAG	Directed acyclic graph.
Eulerian	Graph with a trail visiting each edge exactly once.
Hamiltonian	Graph with a cycle visiting each vertex exactly once.
Cut	A set of edges whose removal increases the number of components.
Cut-set	A minimal cut.
Cut edge	A size 1 cut.
k-Connected	A graph connected with the removal of any $k-1$ vertices.
k-Tough	$\forall S \subseteq V, S \neq \emptyset$ we have $k \cdot c(G - S) \leq \lvert S \rvert$.
k-Regular	A graph where all vertices have degree k.
k-Factor	A k-regular spanning subgraph.
Matching	A set of edges, no two of which are adjacent.
Clique	A set of vertices, all of which are adjacent.
Ind. set	A set of vertices, none of which are adjacent.
Vertex cover	A set of vertices which cover all edges.
Planar graph	A graph which can be embeded in the plane.
Plane graph	An embedding of a planar graph.

Planar graphs

$\sum_{v \in V} \deg(v) = 2m$

If G is planar then $n - m + f = 2$, so

$f \leq 2n - 4, \quad m \leq 3n - 6$

Any planar graph has a vertex with degree ≤ 5.

Notation:

$E(G)$	Edge set
$V(G)$	Vertex set
$c(G)$	Number of components
$G[S]$	Induced subgraph
$\deg(v)$	Degree of v
$\Delta(G)$	Maximum degree
$\delta(G)$	Minimum degree
$\chi(G)$	Chromatic number
$\chi_E(G)$	Edge chromatic number
G^c	Complement graph
K_n	Complete graph
K_{n_1,n_2}	Complete bipartite graph
$\mathrm{r}(k, \ell)$	Ramsey number

Geometry

Projective coordinates:
The triples (x, y, z), not all x, y and z zero.
$\forall c \neq 0 \; (x, y, z) = (cx, cy, cz)$.

Cartesian	Projective
(x, y)	$(x, y, 1)$
$y = mx + b$	$(m, -1, b)$
$x = c$	$(1, 0, -c)$

Distance formula, L_p and L_∞ metric:

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2},$$

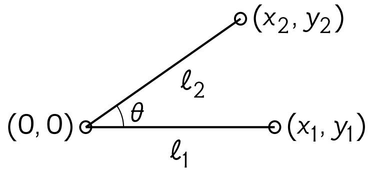
$$\left[|x_1 - x_0|^p + |y_1 - y_0|^p \right]^{1/p},$$

$$\lim_{p \to \infty} \left[|x_1 - x_0|^p + |y_1 - y_0|^p \right]^{1/p}.$$

Area of triangle (x_0, y_0), (x_1, y_1) and (x_2, y_2):

$$\frac{1}{2} \operatorname{abs} \begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{vmatrix}$$

Angle formed by three points:



$$\cos\theta = \frac{(x_1, y_1) \cdot (x_2, y_2)}{\ell_1 \ell_2}$$

Line through two points (x_0, y_0) and (x_1, y_1):

$$\begin{vmatrix} x & y & 1 \\ x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0$$

Area of circle, volume of sphere:
$A = \pi r^2 \quad V = \frac{4}{3}\pi r^3$

Area and volume of a circumscribed cylinder to a sphere:
$A_{cyl} = \frac{3}{2} A_{sph}, \; V_{cyl} = \frac{3}{2} V_{sph}$

Archimedes

If I have seen farther than others, it is because I have stood on the shoulders of giants.

— Issac Newton

Mathematical formulæ and facts

π

Wallis' identity:
$\pi = 2 \cdot \frac{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdots}{1\cdot3\cdot3\cdot5\cdot5\cdot7\cdots}$

Brouncker's continued fraction expansion:
$$\frac{\pi}{4} = 1 + \cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\cdots}}}}$$

Gregrory's series:
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

Newton's series:
$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2\cdot3\cdot2^3} + \frac{1\cdot3}{2\cdot4\cdot5\cdot2^5} + \cdots$$

Sharp's series:
$$\frac{\pi}{6} = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3^1\cdot3} + \frac{1}{3^2\cdot5} - \frac{1}{3^3\cdot7} + \cdots\right)$$

Euler's series:
$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$
$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots$$
$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots$$

Partial Fractions

Let $N(x)$ and $D(x)$ be polynomial functions of x.
We can break down $N(x)/D(x)$ using partial fraction expansion.
First, if the degree of N is greater than or equal to the degree of D, divide N by D, obtaining
$$\frac{N(x)}{D(x)} = Q(x) + \frac{N'(x)}{D(x)}$$
where the degree of N' is less than that of D.

Second, factor $D(x)$
Use the following rules:
For a non-repeated factor:
$$\frac{N(x)}{(x-a)D(x)} = \frac{A}{x-a} + \frac{N'(x)}{D(x)}$$ where
$$A = \left[\frac{N(x)}{D(x)}\right]_{x=a}$$
For a repeated factor:
$$\frac{N(x)}{(x-a)^m D(x)} = \sum_{k=0}^{m-1} \frac{A_k}{(x-a)^{m-k}} + \frac{N'(x)}{D(x)}$$
where
$$A_k = \frac{1}{k!}\left[\frac{d^k}{dx^k}\left(\frac{N(x)}{D(x)}\right)\right]_{x=a}$$

The reasonable man adapts himself to the world; the unreasonable persists in trying to adapt the world to himself. Therefore all progress depends on the unreasonable.

— George Bernard Shaw

Calculus

Derivatives:

1. $\frac{d(cu)}{dx} = c\frac{du}{dx}$ **2.** $\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

3. $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ **4.** $\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}$

5. $\frac{d(u/v)}{dx} = \frac{v(\frac{du}{dx}) - u(\frac{dv}{dx})}{v^2}$ **6.** $\frac{d(e^{cu})}{dx} = ce^{cu}\frac{du}{dx}$

7. $\frac{d(c^u)}{dx} = (\ln c)c^u\frac{du}{dx}$ **8.** $\frac{d(\ln u)}{dx} = \frac{1}{u}\frac{du}{dx}$ **9.** $\frac{d(\sin u)}{dx} = \cos u\frac{du}{dx}$

10. $\frac{d(\cos u)}{dx} = -\sin u\frac{du}{dx}$ **11.** $\frac{d(\tan u)}{dx} = \sec^2 u\frac{du}{dx}$

12. $\frac{d(\cot u)}{dx} = \csc^2 u\frac{du}{dx}$ **13.** $\frac{d(\sec u)}{dx} = \tan u\ \sec u\frac{du}{dx}$

14. $\frac{d(\csc u)}{dx} = -\cot u\ \csc u\frac{du}{dx}$ **15.** $\frac{d(\arcsin u)}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

16. $\frac{d(\arccos u)}{dx} = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$ **17.** $\frac{d(\arctan u)}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$

18. $\frac{d(\text{arccot}\, u)}{dx} = \frac{-1}{1+u^2}\frac{du}{dx}$ **19.** $\frac{d(\text{arcsec}\, u)}{dx} = \frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}$

20. $\frac{d(\text{arccsc}\, u)}{dx} = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx}$ **21.** $\frac{d(\sinh u)}{dx} = \cosh u\frac{du}{dx}$

22. $\frac{d(\cosh u)}{dx} = \sinh u\frac{du}{dx}$ **23.** $\frac{d(\tanh u)}{dx} = \text{sech}^2\, u\frac{du}{dx}$

24. $\frac{d(\coth u)}{dx} = -\text{csch}^2\, u\frac{du}{dx}$ **25.** $\frac{d(\text{sech}\, u)}{dx} = -\text{sech}\, u\ \tanh u\frac{du}{dx}$

26. $\frac{d(\text{csch}\, u)}{dx} = -\text{csch}\, u\ \coth u\frac{du}{dx}$ **27.** $\frac{d(\text{arcsinh}\, u)}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$

28. $\frac{d(\text{arccosh}\, u)}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$ **29.** $\frac{d(\text{arctanh}\, u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}$

30. $\frac{d(\text{arccoth}\, u)}{dx} = \frac{1}{u^2-1}\frac{du}{dx}$ **31.** $\frac{d(\text{arcsech}\, u)}{dx} = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx}$

32. $\frac{d(\text{arccsch}\, u)}{dx} = \frac{-1}{|u|\sqrt{1+u^2}}\frac{du}{dx}$.

Integrals:

1. $\int cu\,dx = c\int u\,dx$ **2.** $\int (u+v)\,dx = \int u\,dx + \int v\,dx$

3. $\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1$ **4.** $\int \frac{1}{x}dx = \ln|x|$

5. $\int e^x\,dx = e^x$ **6.** $\int \frac{dx}{1+x^2} = \arctan x$

7. $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$ **8.** $\int \sin x\,dx = -\cos x$

9. $\int \cos x\,dx = \sin x$ **10.** $\int \tan x\,dx = -\ln|\cos x|$

11. $\int \cot x\,dx = \ln|\cos x|$ **12.** $\int \sec x\,dx = \ln|\sec x + \tan x|$

13. $\int \csc x\,dx = \ln|\csc x + \cot x|$

14. $\int \arcsin\frac{x}{a}dx = \arcsin\frac{x}{a} + \sqrt{a^2-x^2},\ a > 0$

15. $\int \arccos \frac{x}{a}\,dx = \arccos \frac{x}{a} - \sqrt{a^2-x^2},\ a>0$ **16.** $\int \arctan \frac{x}{a}\,dx = x\arctan \frac{x}{a} - \frac{a}{2}\ln(a^2+x^2),\ a>0$

17. $\int \sin^2(ax)\,dx = \frac{1}{2a}\big(ax - \sin(ax)\cos(ax)\big)$ **18.** $\int \cos^2(ax)\,dx = \frac{1}{2a}\big(ax + \sin(ax)\cos(ax)\big)$

19. $\int \sec^2 x\,dx = \tan x$ **20.** $\int \csc^2 x\,dx = -\cot x$ **21.** $\int \sin^n x\,dx = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx$

22. $\int \cos^n x\,dx = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx$ **23.** $\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx,\ n\neq 1$

24. $\int \cot^n x\,dx = -\frac{\cot^{n-1}x}{n-1} - \int \cot^{n-2}x\,dx,\ n\neq 1$

25. $\int \sec^n x\,dx = \frac{\tan x\sec^{n-1}x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx,\ n\neq 1$

26. $\int \csc^n x\,dx = -\frac{\cot x\csc^{n-1}x}{n-1} + \frac{n-2}{n-1}\int \csc^{n-2}x\,dx,\ n\neq 1$ **27.** $\int \sinh x\,dx = \cosh x$

28. $\int \cosh x\,dx = \sinh x$ **29.** $\int \tanh x\,dx = \ln|\cosh x|$ **30.** $\int \coth x\,dx = \ln|\sinh x|$

31. $\int \operatorname{sech} x\,dx = \arctan\sinh x$ **32.** $\int \operatorname{csch} x\,dx = \ln\left|\tanh\frac{x}{2}\right|$ **33.** $\int \sinh^2 x\,dx = \frac{1}{4}\sinh(2x) - \frac{1}{2}x$

34. $\int \cosh^2 x\,dx = \frac{1}{4}\sinh(2x) + \frac{1}{2}x$ **35.** $\int \operatorname{sech}^2 x\,dx = \tanh x$

36. $\int \operatorname{arcsinh}\frac{x}{a}\,dx = x\operatorname{arcsinh}\frac{x}{a} - \sqrt{x^2+a^2},\ a>0$

37. $$\int \operatorname{arccosh}\frac{x}{a}\,dx = \begin{cases} x\operatorname{arccosh}\frac{x}{a} - \sqrt{x^2+a^2}, & \text{if } \operatorname{arccosh}\frac{x}{a}>0 \text{ and } a>0 \\ x\operatorname{arccosh}\frac{x}{a} + \sqrt{x^2+a^2}, & \text{if } \operatorname{arccosh}\frac{x}{a}<0 \text{ and } a>0 \end{cases}$$

38. $\int \operatorname{arctanh}\frac{x}{a}\,dx = x\operatorname{arctanh}\frac{x}{a} + \frac{a}{2}\ln|a^2-x^2|$ **39.** $\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left(x+\sqrt{a^2+x^2}\right),\ a>0$

40. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a},\ a>0$ **41.** $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a},\ a>0$

42. $\int (a^2-x^2)^{3/2}\,dx = \frac{x}{8}(5a^2-2x^2)\sqrt{a^2-x^2} + \frac{3a^4}{8}\arcsin\frac{x}{a},\ a>0$ **43.** $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a},\ a>0$

44. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|$ **45.** $\int \frac{dx}{(a^2-x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2-x^2}}$

46. $\int \sqrt{a^2\pm x^2}\,dx = \frac{x}{2}\sqrt{a^2\pm x^2} \pm \frac{a^2}{2}\ln\left|x+\sqrt{a^2\pm x^2}\right|$ **47.** $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right|,\ a>0$

48. $\int \frac{dx}{ax^2+bx} = \frac{1}{a}\ln\left|\frac{x}{a+bx}\right|$ **49.** $\int x\sqrt{a+bx}\,dx = \frac{2(3bx-2a)(a+bx)^{3/2}}{15b^2}$

50. $\int \frac{\sqrt{a+bx}}{x}\,dx = 2\sqrt{a+bx} + a\int \frac{1}{x\sqrt{a+bx}}\,dx$ **51.** $\int \frac{x}{\sqrt{a+bx}}\,dx = \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right|,\ a>0$

52. $\int \frac{\sqrt{a^2-x^2}}{x}\,dx = \sqrt{a^2-x^2} - a\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$ **53.** $\int x\sqrt{a^2-x^2}\,dx = -\frac{1}{3}(a^2-x^2)^{3/2}$

54. $\int x^2\sqrt{a^2-x^2}\,dx = \frac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2} + \frac{a^4}{8}\arcsin\frac{x}{a},\ a>0$

55. $\int \frac{dx}{\sqrt{a^2-x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$ **56.** $\int \frac{x\,dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}$

57. $\int \frac{x^2\,dx}{\sqrt{a^2-x^2}} = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a},\ a>0$

58. $\int \frac{\sqrt{a^2+x^2}}{x}\,dx = \sqrt{a^2+x^2} - a\ln\left|\frac{a+\sqrt{a^2+x^2}}{x}\right|$ **59.** $\int \frac{\sqrt{x^2-a^2}}{x}\,dx = \sqrt{x^2-a^2} - a\arccos\frac{a}{|x|},\ a>0$

60. $\int x\sqrt{x^2\pm a^2}\,dx = \frac{1}{3}(x^2\pm a^2)^{3/2}$ **61.** $\int \frac{dx}{x\sqrt{x^2+a^2}} = \frac{1}{a}\ln\left|\frac{x}{a+\sqrt{a^2+x^2}}\right|$

Mathematical formulæ and facts

Calculus Cont.

62. $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\arccos\frac{a}{|x|} \qquad a>0$

63. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2\pm a^2}} = \mp\frac{\sqrt{x^2\pm a^2}}{a^2x}$ **64.** $\displaystyle\int \frac{x\,dx}{\sqrt{x^2\pm a^2}} = \sqrt{x^2\pm a^2}$

65. $\displaystyle\int \frac{\sqrt{x^2\pm a^2}}{x^4}\,dx = \mp\frac{(x^2+a^2)^{3/2}}{3a^2x^3}$

66. $$\int \frac{dx}{ax^2+bx+c} = \begin{cases} \dfrac{1}{\sqrt{b^2-4ac}}\ln\left|\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| & \text{if } b^2>4ac \\ \dfrac{2}{\sqrt{4ac-b^2}}\arctan\dfrac{2ax+b}{\sqrt{4ac-b^2}} & \text{if } b^2<4ac \end{cases}$$

67. $$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \begin{cases} \dfrac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right| & \text{if } a>0 \\ \dfrac{1}{\sqrt{-a}}\arcsin\dfrac{-2ax-b}{\sqrt{b^2-4ac}} & \text{if } a<0 \end{cases}$$

68. $$\int \sqrt{ax^2+bx+c}\,dx = \frac{2ax+b}{4a}\sqrt{ax^2+bx+c} + \frac{4ax-b^2}{8a}\int\frac{dx}{\sqrt{ax^2+bx+c}}$$

68. $\displaystyle\int \frac{x\,dx}{\sqrt{ax^2+bx+c}} = \frac{\sqrt{ax^2+bx+c}}{a} - \frac{b}{2a}\int\frac{dx}{\sqrt{ax^2+bx+c}}$

69. $$\int \frac{dx}{x\sqrt{ax^2+bx+c}} = \begin{cases} \dfrac{-1}{\sqrt{c}}\ln\left|\dfrac{2\sqrt{c}\sqrt{ax^2+bx+c}+bx+2c}{x}\right| & \text{if } c>0 \\ \dfrac{1}{\sqrt{-c}}\arcsin\dfrac{bx+2c}{|x|\sqrt{b^2-4ac}} & \text{if } c<0 \end{cases}$$

70. $\displaystyle\int x^3\sqrt{x^2+a^2}\,dx = (\tfrac{1}{3}x^2-\tfrac{2}{15}a^2)(x^2+a^2)^{3/2}$

71. $\displaystyle\int x^n\sin(ax)\,dx = -\frac{1}{a}x^n\cos(ax) + \frac{n}{a}\int x^{n-1}\cos(ax)\,dx$

72. $\displaystyle\int x^n\cos(ax)\,dx = \frac{1}{a}x^n\sin(ax) - \frac{n}{a}\int x^{n-1}\sin(ax)\,dx$

73. $\displaystyle\int x^ne^{ax}\,dx = \frac{x^ne^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$

74. $\displaystyle\int x^n\ln(ax)\,dx = x^{n+1}\left(\frac{\ln(ax)}{n+1} - \frac{1}{(n+1)^2}\right)$

75. $\displaystyle\int x^n(\ln ax)^m\,dx = \frac{x^{n+1}}{n+1}(\ln ax)^m - \frac{m}{n+1}\int x^n(\ln ax)^{m-1}\,dx$

Finite Calculus

Difference, shift operators:

$\Delta f(x) = f(x+1) - f(x)i$
$\mathbb{E}f(x) = f(x+1)$

Fundamental Theorem:

$f(x) = \Delta F(x) \Leftrightarrow \sum f(x)\delta x = F(x)+C$
$\sum_a^b f(x)\delta x = \sum_{i=a}^{b-1} f(i)$

Differences:

$\Delta(cu) = c\Delta u \quad \Delta(u+v) = \Delta u + \Delta v$
$\Delta(uv) = u\Delta v + \mathbb{E}v\Delta u$
$\Delta(x^{\underline{n}}) = nx^{\underline{n-1}} \quad \Delta(H_x) = x^{\underline{-1}}$
$\Delta(2^x) = 2^x \quad \Delta(c^x) = (c-1)c^x$
$\Delta\binom{x}{m} = \binom{x}{m-1}$.

Sums:

$\sum cu\,\delta x = c\sum u\,\delta x$
$\sum(u+v)\,\delta x = \sum u\,\delta x + \sum v\,\delta x$
$\sum u\Delta v\,\delta x = uv - \sum \mathbb{E}v\Delta u\,\delta x$
$\sum x^{\underline{n}}\,\delta x = \frac{x^{\underline{n+1}}}{m+1} \qquad \sum x^{\underline{-1}}\,\delta x = H_x$
$\sum c^x\,\delta x = \frac{c^x}{c-1} \qquad \sum\binom{x}{m}\,\delta x = \binom{x}{m+1}$

Falling Factorial Powers:

$x^{\underline{n}} = x(x-1)\cdots(x-n+1), \quad n>0$
$x^{\underline{0}} = 1 \quad x^{\underline{n}} = \frac{1}{(x+1)\cdots(x+|n|)}, \quad n<0$
$x^{\underline{n+m}} = x^{\underline{m}}(x-m)^{\underline{n}}$

Rising Factorial Powers:

$x^{\overline{n}} = x(x+1)\cdots(x+n-1), \quad n>0$
$x^{\overline{0}} = 1 \quad x^{\overline{n}} = \frac{1}{(x-1)\cdots(x-|n|)}, \quad n<0$
$x^{\overline{n+m}} = x^{\overline{m}}(x+m)^{\overline{n}}$

Conversion:

$x^{\underline{n}} = (-1)^n(-x)^{\overline{n}} = (x-n+1)^{\overline{n}} = 1/(x+1)^{\overline{-n}}$
$x^{\overline{n}} = (-1)^n(-x)^{\underline{n}} = (x+n-1)^{\underline{n}} = 1/(x-1)^{\underline{-n}}$
$x^n = \sum_{k=1}^n \left\{ {n \atop k} \right\} x^{\underline{k}} = \sum_{k=1}^n \left\{ {n \atop k} \right\} (-1)^{n-k}x^{\overline{k}}$
$x^{\underline{n}} = \sum_{k=1}^n \left[{n \atop k} \right] (-1)^{n-k}x^k$
$x^{\overline{n}} = \sum_{k=1}^n \left[{n \atop k} \right] x^k$

$$\begin{aligned}
x^1 &= x^{\underline{1}} &&= x^{\overline{1}}\\
x^2 &= x^{\underline{2}} + x^{\underline{1}} &&= x^{\overline{2}} - x^{\overline{1}}\\
x^3 &= x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}} &&= x^{\overline{3}} - 3x^{\overline{2}} + x^{\overline{1}}\\
x^4 &= x^{\underline{4}} + 6x^{\underline{3}} + 7x^{\underline{2}} + x^{\underline{1}} &&= x^{\overline{4}} - 6x^{\overline{3}} + 7x^{\overline{2}} - x^{\overline{1}}\\
x^5 &= x^{\underline{5}} + 15x^{\underline{4}} + 25x^{\underline{3}} + 10x^{\underline{2}} + x^{\underline{1}} &&= x^{\underline{5}} - 15x^{\overline{4}} + 25x^{\overline{3}} - 10x^{\overline{2}} + x^{\overline{1}}
\end{aligned}$$

$$\begin{aligned}
x^{\overline{1}} &= x^1 & x^{\underline{1}} &= x^1\\
x^{\overline{2}} &= x^2 + x^1 & x^{\underline{2}} &= x^2 - x^1\\
x^{\overline{3}} &= x^3 + 3x^2 + 2x^1 & x^{\underline{3}} &= x^3 - 3x^2 + 2x^1\\
x^{\overline{4}} &= x^4 + 6x^3 + 11x^2 + 6x^1 & x^{\underline{4}} &= x^4 - 6x^3 + 11x^2 - 6x^1\\
x^{\overline{5}} &= x^5 + 10x^4 + 35x^3 + 50x^2 + 24x^1 & x^{\underline{5}} &= x^5 - 10x^4 + 35x^3 - 50x^2 + 24x^1
\end{aligned}$$

Aus dem Paradies, das Cantor uns geschaffen, soll uns niemand vertreiben können.

— David Hilbert

From the paradise, that Cantor created for us, no-one shall be able to expel us.

Mathematical formulæ and facts

Series

Taylor's series centered at a:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \cdots = \sum_{i=0}^{\infty} \frac{(x-a)^i}{i!} f^{(i)}(a)$$

Expansions:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{i=0}^{\infty} x^i$$

$$\frac{1}{1-cx} = 1 + cx + c^2x^2 + c^3x^3 + \cdots = \sum_{i=0}^{\infty} c^i x^i$$

$$\frac{1}{1-x^n} = 1 + x^n + x^{2n} + x^{3n} + \cdots = \sum_{i=0}^{\infty} x^{ni}$$

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots = \sum_{i=0}^{\infty} i x^i$$

$$\sum_{k=0}^{n} \left\{ {n \atop k} \right\} \frac{k! z^k}{(1-z)^{k+1}} = x + 2^n x^2 + 3^n x^3 + 4^n x^4 + \cdots = \sum_{i=0}^{\infty} i^n x^i$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 - \cdots = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{x^i}{i}$$

$$\ln \frac{1}{1-x} = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots = \sum_{i=1}^{\infty} \frac{x^i}{i}$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!}$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!}$$

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{2i+1}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{n}{i} x^i$$

$$\frac{1}{(1-x)^{n+1}} = 1 + (n+1)x + \binom{n+2}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{i+n}{i} x^i$$

$$\frac{x}{e^x-1} = 1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \cdots = \sum_{i=0}^{\infty} \frac{B_i x^i}{i!}$$

$$\frac{1}{2x}(1 - \sqrt{1-4x}) = 1 + x + 2x^2 + 5x^3 + \cdots = \sum_{i=0}^{\infty} \frac{1}{i+1}\binom{2i}{i} x^i$$

$$\frac{1}{\sqrt{1-4x}} = 1 + 2x + 6x^2 + 20x^3 + \cdots = \sum_{i=0}^{\infty} \binom{2i}{i} x^i$$

$$\frac{1}{\sqrt{1-4x}} \left(\frac{1-\sqrt{1-4x}}{2x} \right)^n = 1 + (2+n)x + \binom{4+n}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{2i+n}{i} x^i$$

$$\frac{1}{1-x} \ln \frac{1}{1-x} = x + \frac{3}{2}x^2 + \frac{11}{6}x^3 + \frac{25}{12}x^4 + \cdots = \sum_{i=1}^{\infty} H_i x^i$$

$$\frac{1}{2}\left(\ln \frac{1}{1-x} \right)^2 = \frac{1}{2}x^2 + \frac{3}{4}x^3 + \frac{11}{24}x^4 + \cdots = \sum_{i=2}^{\infty} \frac{H_{i-1} x^i}{i}$$

$$\frac{x}{1-x-x^2} = x + x^2 + 2x^3 + 3x^4 + \cdots = \sum_{i=0}^{\infty} F_i x^i$$

$$\frac{F_n x}{1-(F_{n-1}+F_{n+1})x-(-1)^n x^2} = F_n x + F_{2n}x^2 + F_{3n}x^3 + \cdots = \sum_{i=0}^{\infty} F_{ni} x^i.$$

Ordinary power series:

$$A(x) = \sum_{i=0}^{\infty} a_i x^i$$

Exponential power series:

$$A(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}$$

Dirichlet power series: $A(x) = \sum_{i=1}^{\infty} \frac{a_i}{i^x}$

Binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Difference of like powers:

$$x^n - y^n = (x-y) \sum_{k=0}^{n-1} x^{n-1-k} y^k$$

For ordinary power series:

$$\alpha A(x) + \beta B(x) = \sum_{i=0}^{\infty} (\alpha a_i + \beta b_i) x^i$$

$$x^k A(x) = \sum_{i=k}^{\infty} a_{i-k} x^i$$

$$\frac{A(x) - \sum_{i=0}^{k-1} a_i x^i}{x^k} = \sum_{i=0}^{\infty} a_{i+k} x^i$$

$$A(cx) = \sum_{i=0}^{\infty} c^i a_i x^i$$

$$A'(x) = \sum_{i=0}^{\infty} (i+1) a_{i+1} x^i$$

$$xA'(x) = \sum_{i=1}^{\infty} i a_i x^i$$

$$\int A(x)\,dx = \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} x^i$$

$$\frac{A(x)+A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i} x^{2i}$$

$$\frac{A(x)-A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i+1} x^{2i+1}$$

Summation:

If $b_i = \sum_{j=0}^{i} a_i$ then $B(x) = \frac{1}{1-x} A(x)$

Convolution:

$$A(x)B(x) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} a_j b_{i-j} \right) x^i$$

Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk.

— Leopold Kronecker

God made the natural numbers; all the rest is the work of man.

Mathematical formulæ and facts

Series

Expansions:

$$\frac{1}{(1-x)^{n+1}} \ln \frac{1}{1-x} = \sum_{i=0}^{\infty} (H_{n+i} - H_n) \binom{n+i}{i} x^i$$

$$x^{\overline{n}} = \sum_{i=0}^{\infty} \left[{n \atop i} \right] x^i$$

$$\left(\ln \frac{1}{1-x} \right)^n = \sum_{i=0}^{\infty} \left[{i \atop n} \right] \frac{n! x^i}{i!}$$

$$\tan x = \sum_{i=1}^{\infty} (-1)^{i-1} \frac{2^{2i}(2^{2i}-1) B_{2i} x^{2i-1}}{(2i)!}$$

$$\frac{1}{\zeta(x)} = \sum_{i=1}^{\infty} \frac{\mu(i)}{i^x}$$

$$\left(\frac{1}{x} \right)^{\overline{-n}} = \sum_{i=0}^{\infty} \left\{ {i \atop n} \right\} x^i$$

$$(e^x - 1)^n = \sum_{i=0}^{\infty} \left\{ {i \atop n} \right\} \frac{n! x^i}{i!}$$

$$x \cot x = \sum_{i=0}^{\infty} \frac{(-4)^i B_{2i} x^{2i}}{(2i)!}$$

$$\zeta(x) = \sum_{i=1}^{\infty} \frac{1}{i^x}$$

$$\frac{\zeta(x-1)}{\zeta(x)} = \sum_{i=1}^{\infty} \frac{\phi(i)}{i^x}$$

$$\zeta(x) = \prod_p \frac{1}{1-p^{-x}}$$

$$\zeta^2(x) = \sum_{i=1}^{\infty} \frac{d(i)}{x^i} \quad \text{where } d(n) = \sum_{d|n} 1$$

$$\zeta(x)\zeta(x-1) = \sum_{i=1}^{\infty} \frac{S(i)}{x^i} \quad \text{where } S(n) = \sum_{d|n} d$$

$$\zeta(2n) = \frac{2^{2n-1}|B_{2n}|}{(2n)!} \pi^{2n}, \quad n \in \mathbb{N}$$

$$\frac{x}{\sin x} = \sum_{i=0}^{\infty} (-1)^{i-1} \frac{(4^i - 2) B_{2i} x^{2i}}{(2i)!}$$

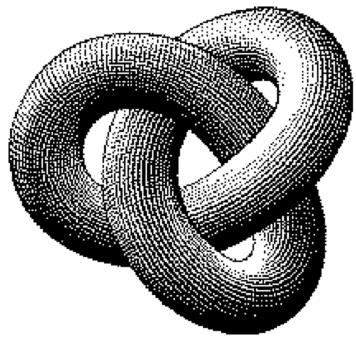
$$\left(\frac{1-\sqrt{1-4x}}{2x} \right)^n = \sum_{i=0}^{\infty} \frac{n(2i+n-1)!}{i!(n+i)!} x^i$$

$$e^x \sin x = \sum_{i=1}^{\infty} \frac{2^{i/2} \sin \frac{i\pi}{4}}{i!} x^i$$

$$\frac{\sqrt{1-\sqrt{1-x}}}{x} = \sum_{i=0}^{\infty} \frac{(4i)!}{16^i \sqrt{2} (2i)! (2i+1)!} x^i$$

$$\left(\frac{\arcsin x}{x} \right)^2 = \sum_{i=0}^{\infty} \frac{4^i i!^2}{(i+1)(2i+1)!} x^{2i}$$

Escher's Knot



Stieltjes Integration

If G is continuous in the interval $[a,b]$ and F is nondecreasing then

$$\int_a^b G(x)\,dF(x)$$

exists.

If $a \le b \le c$ then

$$\int_a^c G(x)\,dF(x) = \int_a^b G(x)\,dF(x) + \int_b^c G(x)\,dF(x)$$

If the integrals involved exist

$$\int_a^b \big(G(x) + H(x)\big)\,dF(x) = \int_a^b G(x)\,dF(x) + \int_a^b H(x)\,dF(x)$$

$$\int_a^b G(x)\,d\big(F(x) + H(x)\big) = \int_a^b G(x)\,dF(x) + \int_a^b G(x)\,dH(x)$$

$$\int_a^b c \cdot G(x)\,dF(x) = \int_a^b G(x)\,d\big(c \cdot F(x)\big) = c \int_a^b G(x)\,dF(x)$$

$$\int_a^b G(x)\,dF(x) = G(b)F(b) - G(a)F(a) - \int_a^b F(x)\,dG(x)$$

If the integrals involved exist, and F possesses a derivative F' at every point in $[a,b]$ then

$$\int_a^b G(x)\,dF(x) = \int_a^b G(x)F'(x)\,dx$$

Cramer's rule

If we have equations:

$$\begin{matrix} a_{1,1}x_1 + a_{1,2}x_2 & \cdots & + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 & \cdots & + a_{2,n}x_n = b_2 \\ & \vdots & \\ a_{n,1}x_1 + a_{n,2}x_2 & \cdots & + a_{n,n}x_n = b_n \end{matrix}$$

Let $A = (a_{i,j})$ and B be the column matrix (b_i). Then there is a unique solution iff $\det A \ne 0$. Let A_i be A with column i replaced by B. Then $x_i = \dfrac{\det A_i}{\det A}$.

The Fibonacci numbers

The Fibonacci number system:
Every integer n has a unique representation

$$n = F_{k_1} + F_{k_2} + \cdots + F_{k_m}$$

where $k_i \ge k_{i+1} + 2$ for all i, $1 \le i < m$ and $k_m \ge 2$.

The first Fibonacci numbers:
1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Additive rule:

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$$
$$F_{2n} = F_n F_{n+1} + F_{n-1} F_n$$

Definitions:

$$F_0 = F_1 = 1$$
$$F_i = F_{i-1} + F_{i-2}$$
$$F_{-i} = (-1)^{i-1}$$
$$\phi = \frac{1+\sqrt{5}}{2}, \quad \hat{\phi} = \frac{1-\sqrt{5}}{2} = 1 - \phi$$
$$F_i = \frac{1}{\sqrt{5}} \left(\phi^i - \hat{\phi}^i \right)$$

Cassini's identity for $i > 0$:

$$F_{i+1}F_{i-1} - F_i^2 = (-1)^i$$

Calculation by matrices:

$$\begin{pmatrix} F_{n-2} & F_{n-1} \\ F_{n-1} & F_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n$$

Improvement makes strait roads, but the crooked roads without Improvement, are roads of Genius.

— William Blake (The Marriage of Heaven and Hell)

The magic square

00	47	18	76	29	93	85	34	61	52
86	11	57	28	70	39	94	45	02	63
95	80	22	67	38	71	49	56	13	04
59	96	81	33	07	48	72	60	24	15
73	69	90	82	44	17	58	01	35	26
68	74	09	91	83	55	27	12	46	30
37	08	75	19	92	84	66	23	50	41
14	25	36	40	51	62	03	77	88	99
21	32	43	54	65	06	10	89	97	78
42	53	64	05	16	20	31	98	79	87