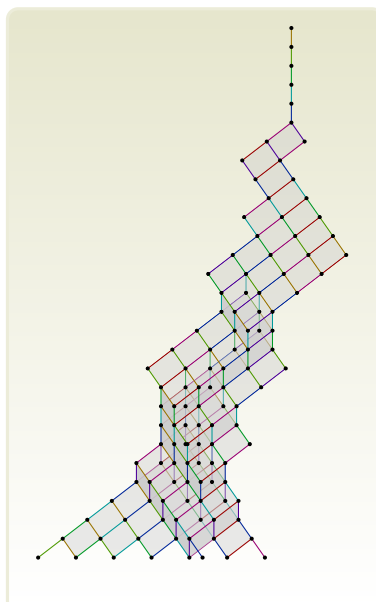
THE LIE HASSE PACKAGE
VERSION 1.02

BENJAMIN McKAY



Contents

1. Quick introduction

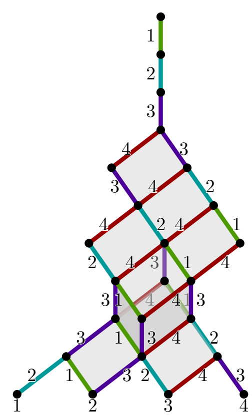
This package draws the Hasse diagram of the poset of the positive simple roots of each complex simple Lie group, as drawn by Ringel [3].

Date: 4 December 2024.

Load the package

```
\documentclass{article}
\usepackage{lie-hasse}
\begin{document}
The Hasse diagram of \(F_4\) is
\begin{center}
\hasse F4
\end{center}
\end{document}
```

The Hasse diagram of F_4 is

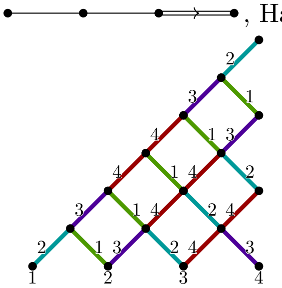


Each edge is labelled with the simple root by which vertices differ.

Inside a Ti*k*Z statement

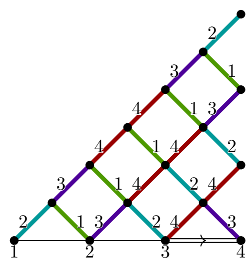
```
\(B_4\) has Dynkin diagram \tikz \dynkin{B}{4};, Hasse diagram
\begin{center}
\hasse[edge length=1cm]B4
\end{center}
```

B_4 has Dynkin diagram , Hasse diagram



Inside a Dynkin diagram environment, diagrams fit together

```
The Hasse diagram of \(B_4\) is
\begin{dynkinDiagram}[vertical shift=0,edge length=1cm]{B}{4}
\hasse[edge length=1cm]B4
\end{dynkinDiagram}
```

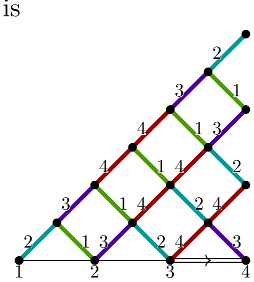
The Hasse diagram of B_4 is 

We shut off the default vertical shift of the Dynkin diagram, so that it starts at the origin. There is an option to `\hasse` for this:

Attaching the Dynkin diagram

```
The Hasse diagram of \(B_4\) is
\begin{center}
\hasse[attach Dynkin diagram=true]B4
\end{center}
```

The Hasse diagram of B_4 is

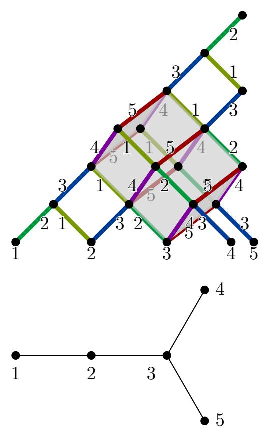


Unfortunately, attaching a Dynkin diagram looks terrible for D or E series, so a Dynkin diagram appears below.

Attaching the Dynkin diagram

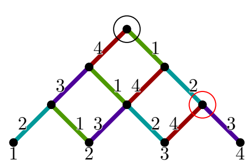
```
The Hasse diagram of \(D_5\) is
\begin{center}
\hasse[attach Dynkin diagram=true]D5
\end{center}
```

The Hasse diagram of D_5 is



Inside a TikZ environment

```
\begin{tikzpicture}
\hasse A4
\draw (4;1) circle (5pt);
\draw[red] (2;3) circle (5pt);
\end{tikzpicture}
```



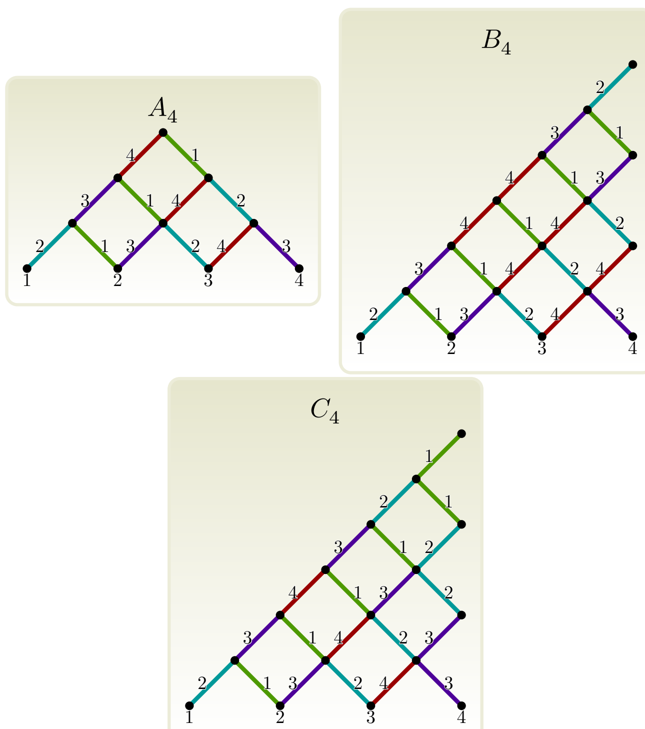
In this example, we see that the roots of the Hasse diagram are Ti*k*Z nodes labelled $g;i$ for grade g (i.e. g units up the page) and index i (i.e. i^{th} root of grade g drawn on the page, starting from the left).

2. Prettier

The package includes a more elaborate `\hasseDiagrams` command, taking a list of semicolon separated Dynkin diagram identfiers.

With some global options to make prettier diagrams

```
\tikzset{
  background rectangle/.style={
    shade,
    top color=olive!20,
    bottom color=white,
    draw=olive!15,
    very thick,
    rounded corners},
  /Lie Hasse diagram,
  edge length=1.2cm,
  show name=true,
  vertical shift=0}
\hasseDiagrams{A4;B4;C4}
```



Global options:

```
\tikzset{/Lie Hasse,
   edge/.style={ultra thick},
   edge quotes/.style={
      /Dynkin diagram/text style,
```

```
auto,
inner sep=2pt},
```

allow to change the edges, and to change the way that labels are printed, and how close labels are to the edges. To give even greater control, the user can define a command `\edgeQuote` of one argument, which is applied to the label placed on every edge of the Hasse diagram. The default is

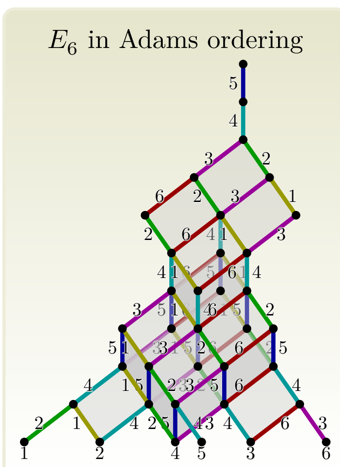
```
\NewDocumentCommand\edgeQuote{m}{\contour{white}{#1}}
```

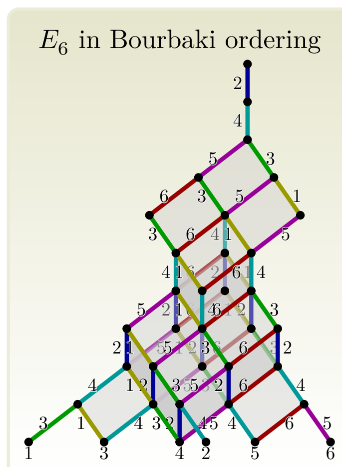
which ensures that every label, which is black text by default, has a slight white outline to make it easier to read against a coloured background.

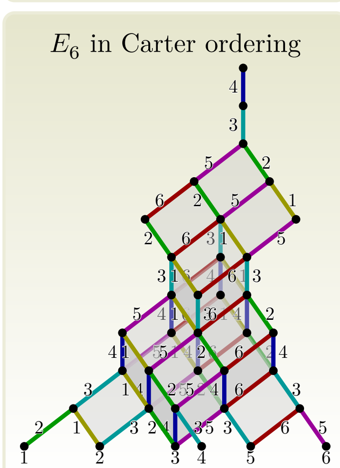
3. Root order

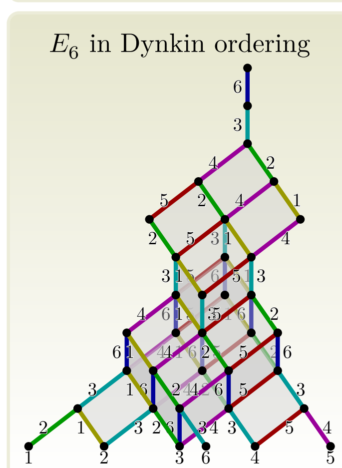
We order the roots as in the Dynkin diagram package: with orderings Adams, Bourbaki, Carter, Dynkin and Kac. *Warning:* the default is Carter, *not* Bourbaki; the default in the Dynkin diagram package is Bourbaki. We can use this like:

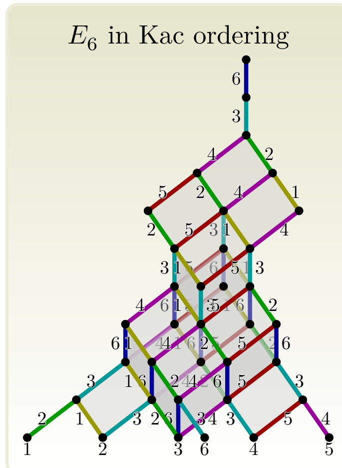
```
\tikzset{/Lie Hasse diagram,show name=true,show ordering=true}
\hasseDiagrams{[ordering=Adams]E6;[ordering=Bourbaki]E6}
\hasseDiagrams{[ordering=Carter]E6;[ordering=Dynkin]E6}
\hasseDiagrams{[ordering=Kac]E6}
```











The Lie Hasse package inherits the ordering of roots from the Dynkin diagrams package, so we can set it with

```
\tikzset{/Dynkin diagram/ordering=Bourbaki}
```

4. Graph height and width

The *height* of a Hasse diagram is the number of grades. The *width* of each grade is the number of vertices on that grade. We recover these with

```
\newcount\h
\rootSystemHeight[G][2]{\h}
```

to store the height of G_2 in a counter called `\h`, and

```
\newcount\w
\rootSystemWidthAtGrade[G][2]{3}{\w}%
```

to store the width of G_2 at grade 3 in a counter called `\w`.

Once you use `\dynkin G2` or `\hasse G2` or the other commands, like

```
\rootSystemHeight[G][2]{\h}
```

the system stores that your default root system is G_2. Subsequently calls to

```
\rootSystemHeight{\h}
```

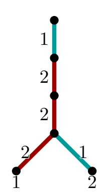
and

```
\rootSystemWidthAtGrade{3}{\w}
```

do not need to specify the root system.

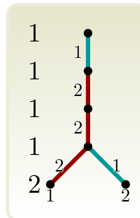
The `show height` option:

```
\tikzset{/Lie Hasse diagram,show name=true,show height=true}
\hasseDiagrams{G2}
```



The `show widths` option:

```
\tikzset{/Lie Hasse diagram/show widths=true}
\hasseDiagrams{G2}
```



5. Root decompositions

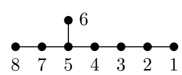
Each positive root in a root system is a unique nonnegative integer linear combination of positive simple roots. We can recover this expression as

```
\rootSum[G][2]{5}{1}{\rs}
```

which, for the root system G_2, and the root at position $5;1$ in our Hasse diagram, stores in the variable `\rs` a string which looks like `2,3`. This is a comma separated list of the integer coefficients. *Warning:* for the moment, this list of coefficients is in Carter ordering. If we omit `[G][2]`, the current default root system is implied.

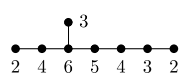
Here is the Dynkin diagram of E_8, indicating the order of the roots in Carter ordering.

```
\dynkin[label,ordering=Carter,edge length=.35cm]{E}{8}
```



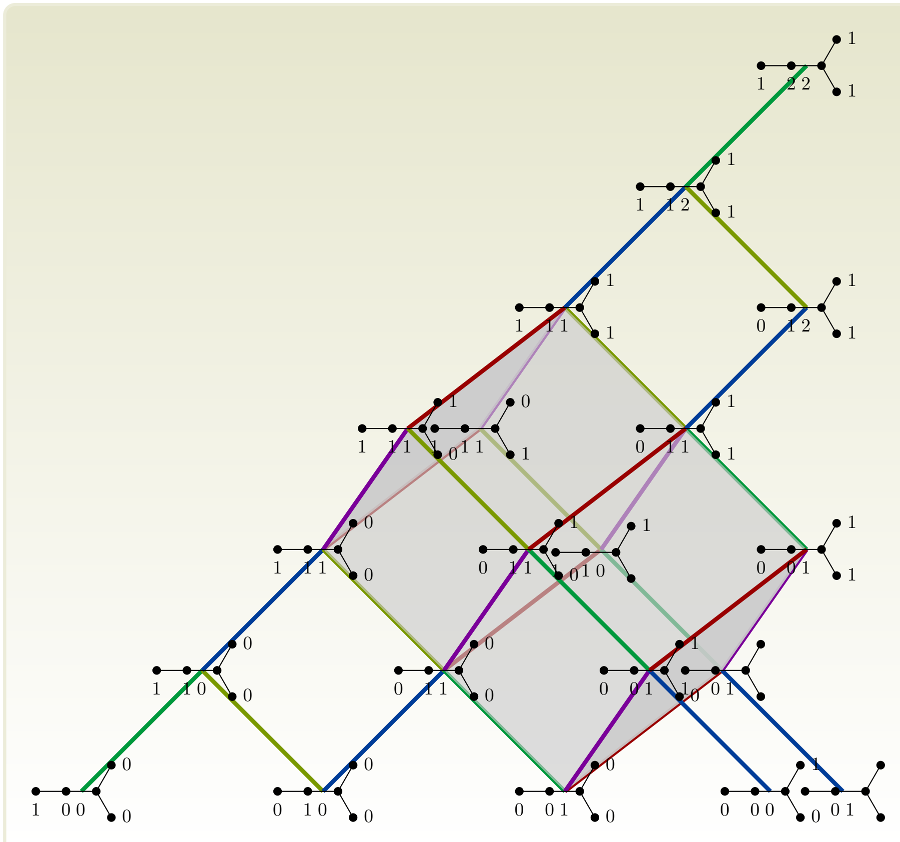
Here is the same Dynkin diagram, except showing, at each simple root, the coefficient of that simple root in the highest root.

```
\rootSum[E][8]{29}{1}{\rs}
\dynkin[expand labels=\rs,ordering=Carter,edge length=.35cm]{E}{8}
```



The option `for all roots` allows execution of code once on every root.

```
\tikzset{/Lie Hasse diagram,
  edge length=3.2cm,
  compact root/.code={},
  noncompact root/.code={},
  edge quotes/.style={opacity=0},
  embedded Dynkin diagram/.style={
    edge length=.4cm,
    root radius=.05cm
  },
  for all roots/.code 2 args={\drawRootAsDynkinSum{#1}{#2}}}
\hasse D5
```



See more below on compact versus noncompact roots; the code `compact` is applied to draw all of the compact roots, and the code `noncompact` to draw the noncompact roots. Setting those codes to be empty, and setting `edge quotes` to be transparent, we get a much simpler Hasse diagram, so that we can see the embedded Dynkin diagrams more clearly.

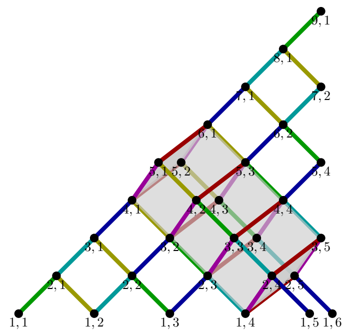
6. For all roots . . .

You can make your own macros loop over all of the roots: you define a macro `\foo{g}{i}`, which is fed the grade g of each root in the diagram, and the *index* i. A simple example:

```
\newcommand{\foo}[2]%
{%
  \node[below,scale=.5] at (#1;#2) {\(#1,#2\)};%
}%
```

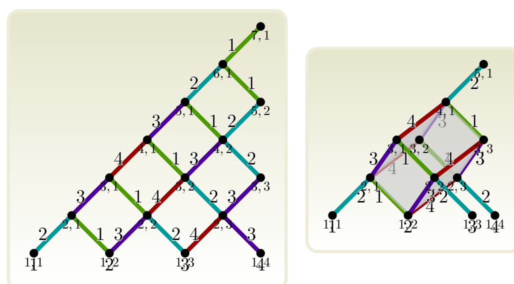
Inside a Ti*k*Z or `dynkinDiagram` environment:

```
\tikzset{/Lie Hasse diagram/edge quotes/.style={opacity=0},
  /Dynkin diagram/edge length=1.5cm}
\begin{tikzpicture}
  \hasse D6%
  \forAllPositiveRoots{\foo}%
\end{tikzpicture}
```



If you put this into the for all roots option, it executes on its own:

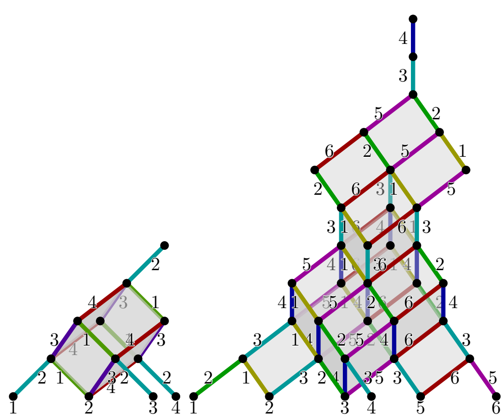
```
\tikzset{/Lie Hasse diagram/for all roots/.code 2 args={\foo{#1}{#2}}}
\hasseDiagrams{C4;D4}
```



7. Three dimensional effect

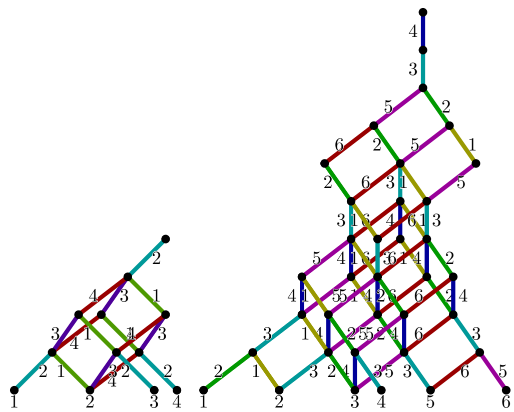
We draw the D, E, F Hasse diagrams, following Ringel [3], as an arrangement of cubes. Nutma [2] draws the Hasse diagrams using a more elementary approach, but including also the affine Kac–Moody algebras. Opposite sides of any square have the same edge label, by commutativity of addition. Hence we don't need to see every edge perfectly. The three dimensional effect is the default:

```
\hasse D4\hasse E6
```



We can turn it off:

```
\hasse[three D=false] D4
\hasse[three D=false] E6
```

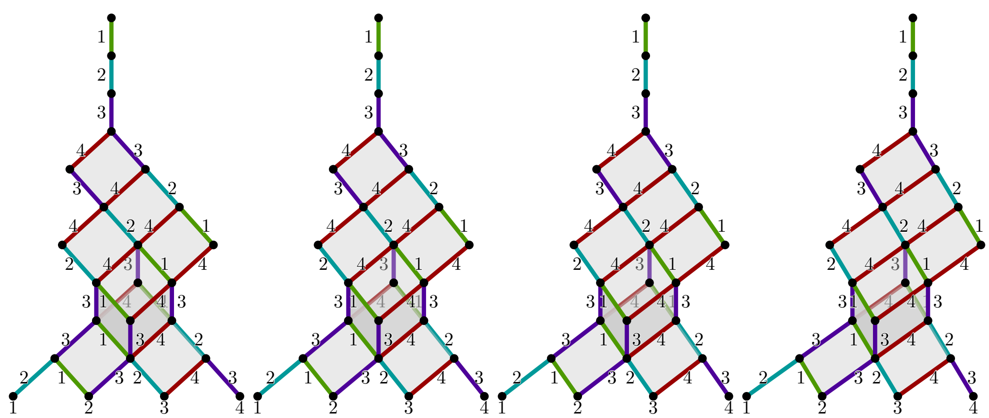


or globally with \tikzset{/Lie Hasse diagram/three D=false}.

The astute reader will perhaps notice that the three dimensional effect is not realistic. To be Hasse diagrams, the roots have to line up horizontally by grade. This is inconsistent with three dimensional projection of our cubes. We have also tried to use only a small number of layers in the three dimensional geometry, so the images are not perfect, but easy enough to read.

We can change the z shift to slant the three dimensional images to the right:

```
\hasse[z shift=.1]F4\hasse[z shift=.2]F4\hasse[z shift=.3]F4\hasse[z shift=.4]F4
```

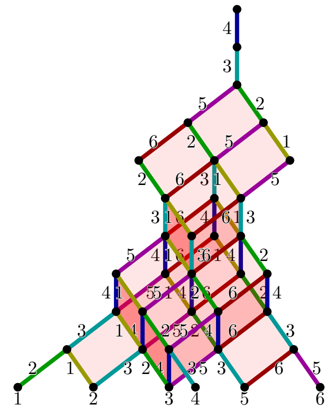


We only use three colours and opacities for the faces:

```
  top/.style={black!20,opacity=.4},
  left/.style={black!20,opacity=.9},
  right/.style={black!20,opacity=.6},
```

You can change these:

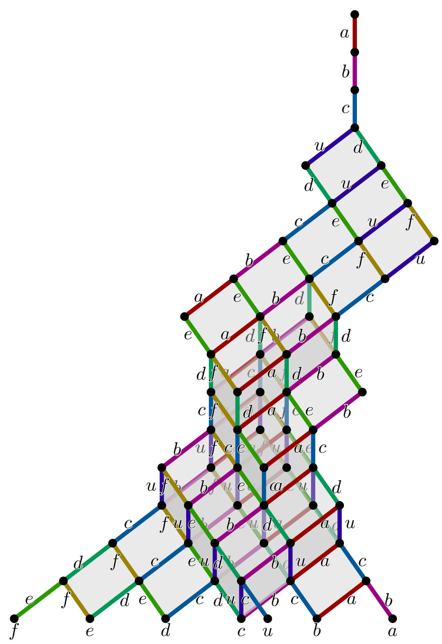
```
\hasse[
  top/.style={red,opacity=.1},
  right/.style={red,opacity=.2},
  left/.style={red,opacity=.4}]E6
```



8. Label the simple roots

Ringel [3] labels his edges like

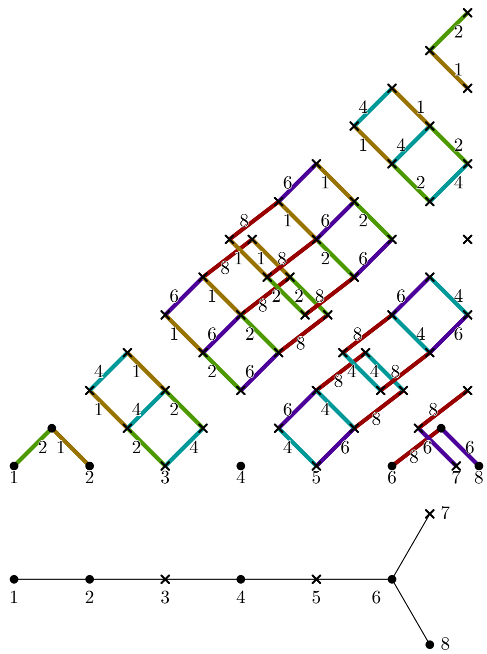
```
\hasse[labels={f,e,d,c,u,b,a}]E7
```



9. Parabolic subgroups

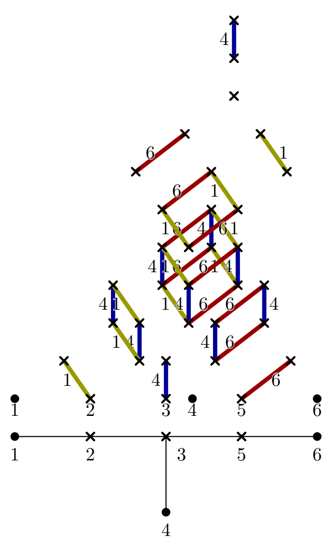
This package offers nothing over Ringel's original pictures, except that the user can pick some simple roots whose associated edges are drawn differently. The chosen simple roots are called *compact*, following terminology from the theory of parabolic subgroups. We let the reader explore the notation for parabolic subgroups in the Dynkin diagrams package, and use this to declare various roots compact.

```
\tikzset{/Lie Hasse diagram,attach Dynkin diagram=true,three D=false}
\hasse D{**x*x*x*}
```

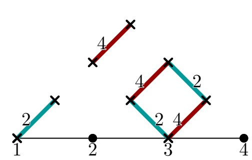


Our motivation comes from trying to identify the invariant vector subbundles of the tangent bundle of a rational homogeneous variety [1]. Such diagrams are often unreadable if we don't turn off the three dimensional graphics. By default, noncompact root edges are not drawn.

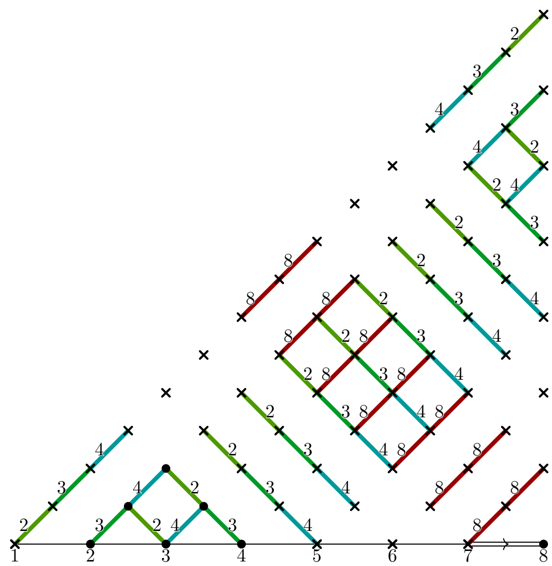
```
\hasse E{*xx*x*}
```



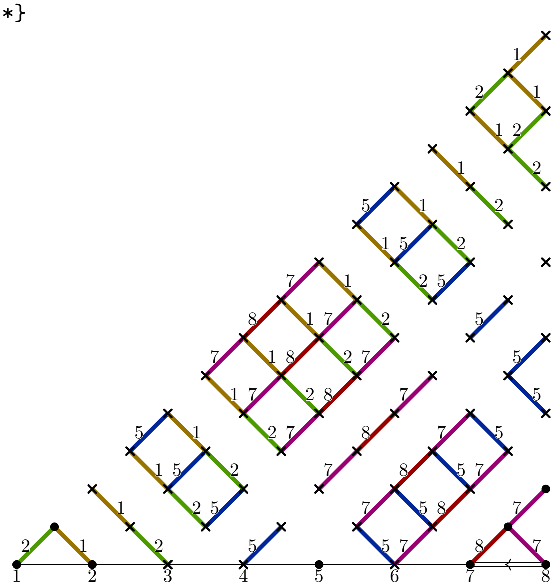
```
\hasse A{x*x*}
```



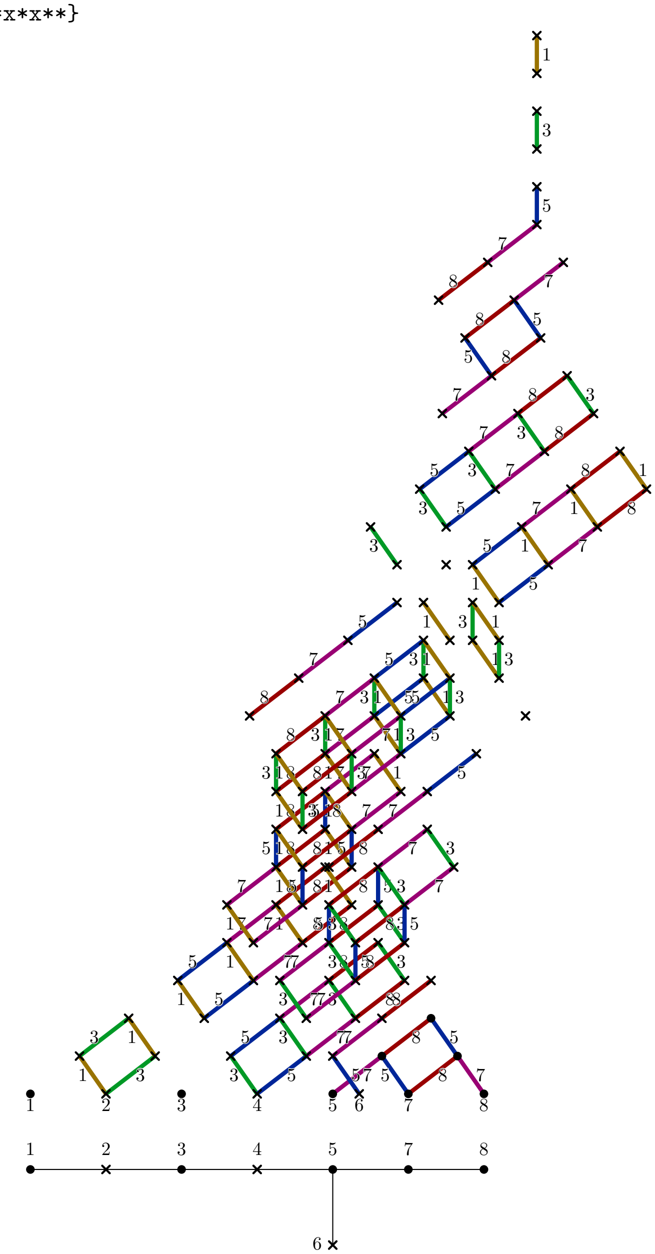
\hasse[parabolic=113]B8



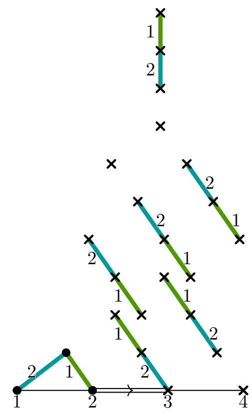
\hasse C{**xx*x**}



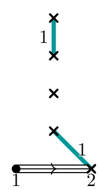
`\hasse E{*x*x*x**}`



\hasse F{**xx}

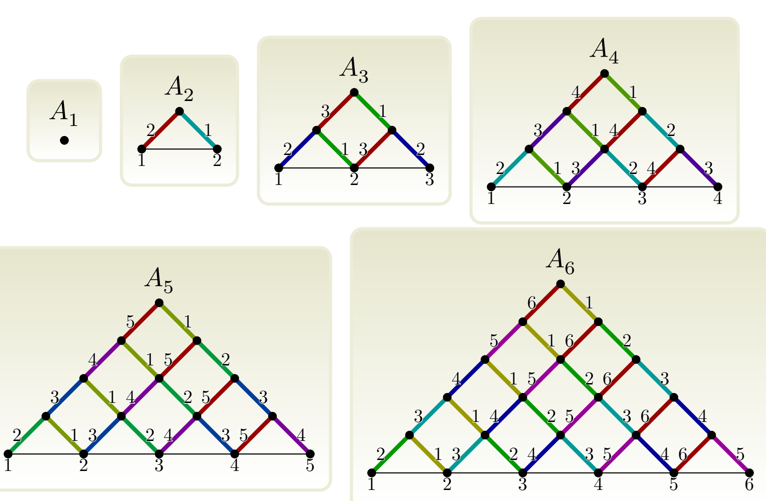


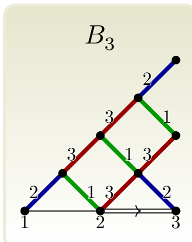
\hasse G{*x}

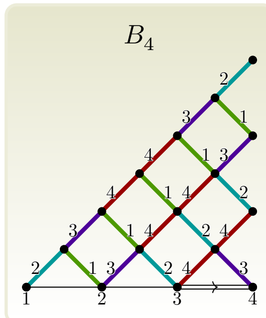


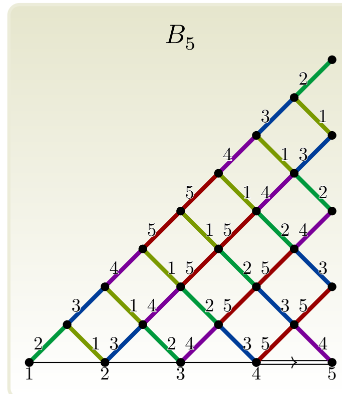
10. Examples

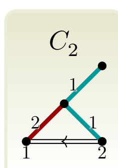
\hasseDiagrams{A1;A2;A3;A4;A5;A6}
\hasseDiagrams{B3;B4;B5}
\hasseDiagrams{C2;C3;C4}
\hasseDiagrams{C5;C6}
\hasseDiagrams{E6;E7}
\hasseDiagrams{E8}
\hasseDiagrams{F4;G2}

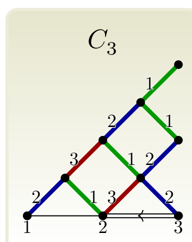


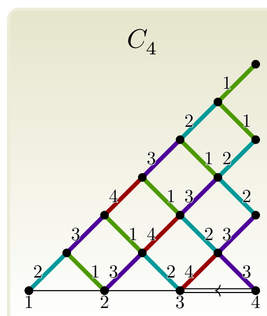

B_3

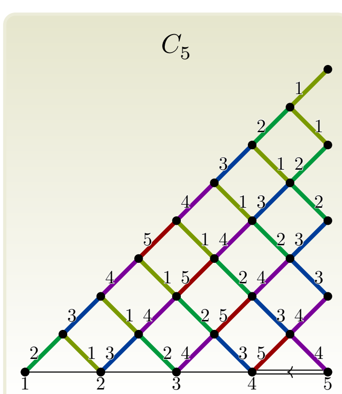

B_4

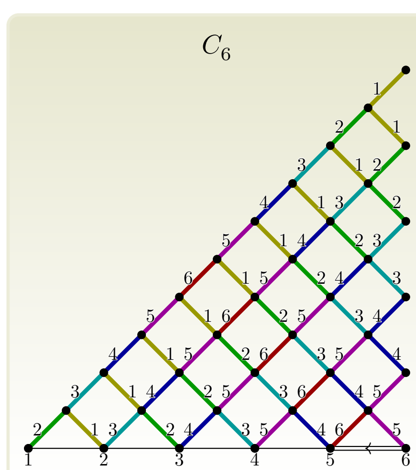

B_5

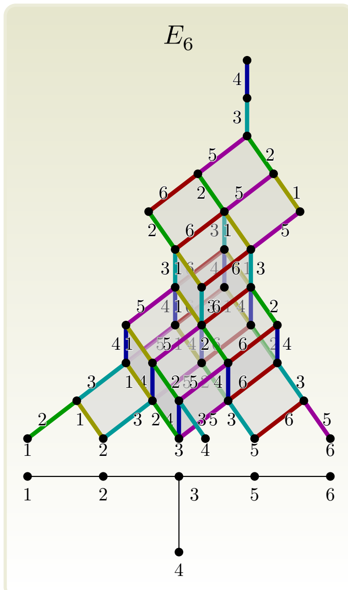

C_2

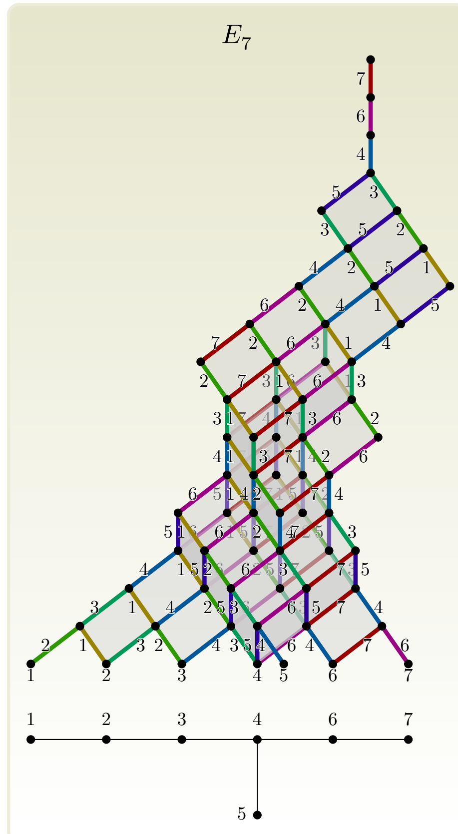

C_3

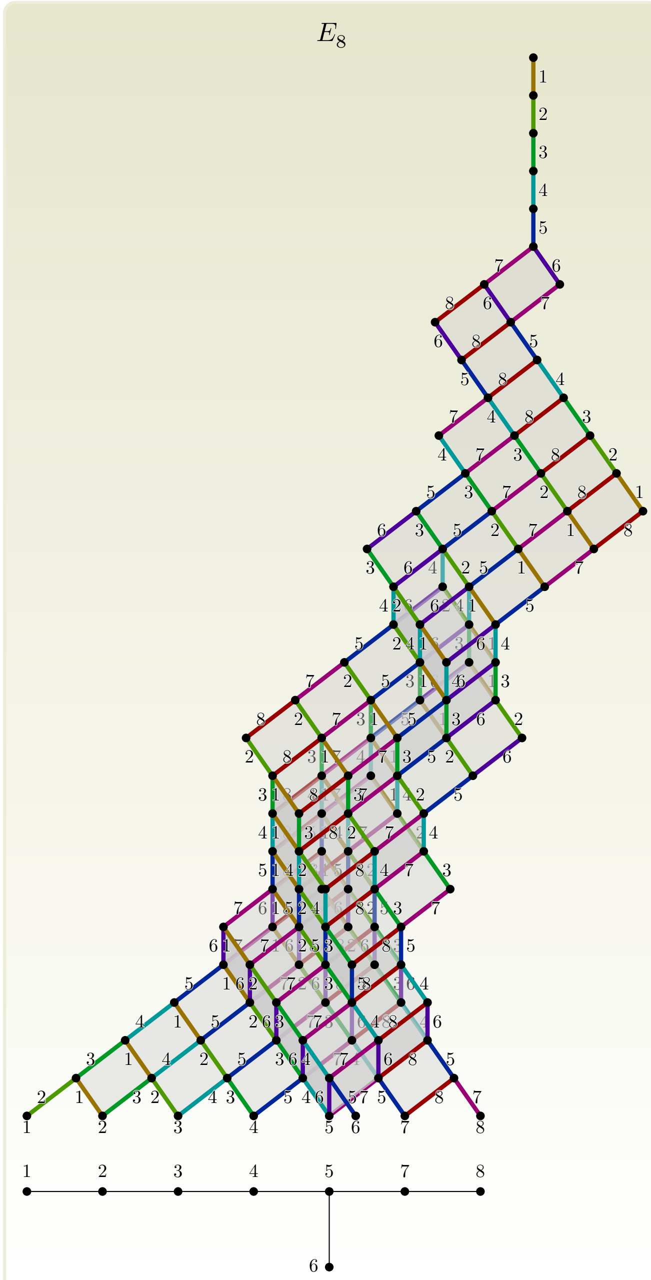

C_4

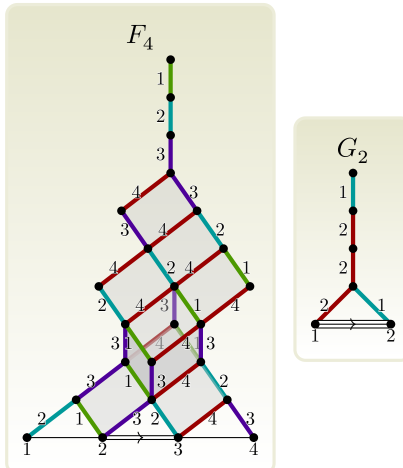

C_5


C_6


E_6


E_7


E_8
1
2
3
4
5
7
8
6

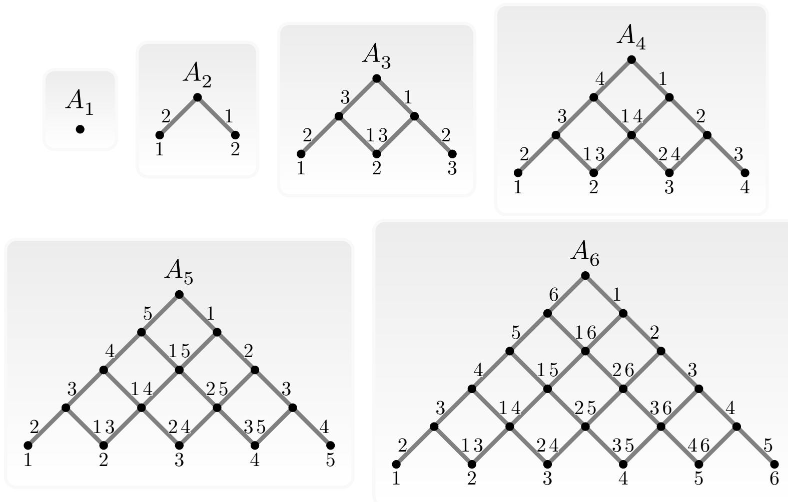


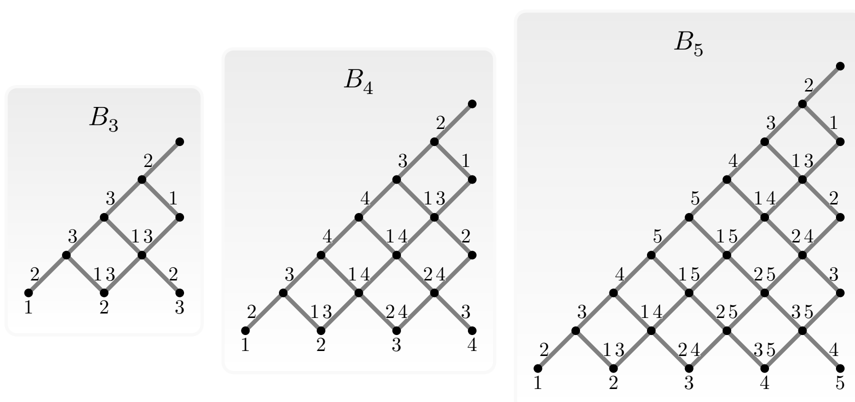
11. Black and white

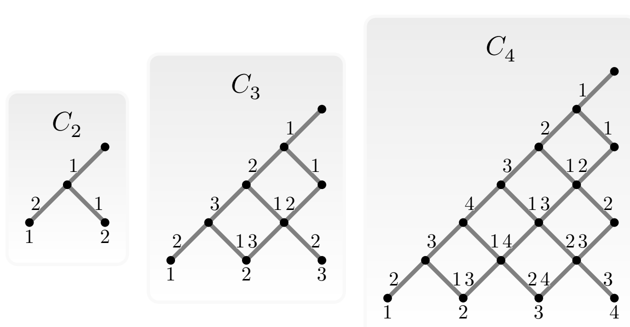
Publishing in colour on paper can be expensive. Simple global options:

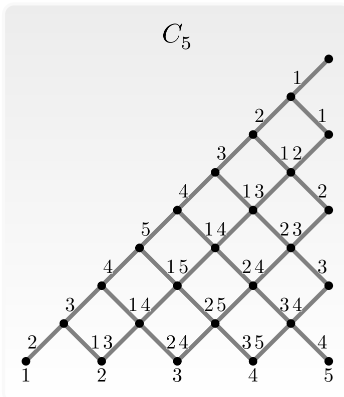
```
\tikzset{
  background rectangle/.style={
    shade,
    top color=gray!15,
    bottom color=white,
    draw=gray!5,
    very thick,
    rounded corners},
  /Dynkin diagram/text style/.style={black,scale=.75},
  /Lie Hasse diagram,
  edge length=1cm,
  edge/.style={draw=black!50,ultra thick},
  edge quotes/.style={black,auto,inner sep=3pt,scale=.75},
  three D=true,
  show name=true}
```

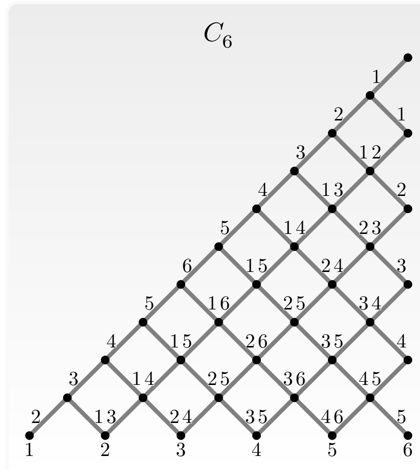
change our examples to

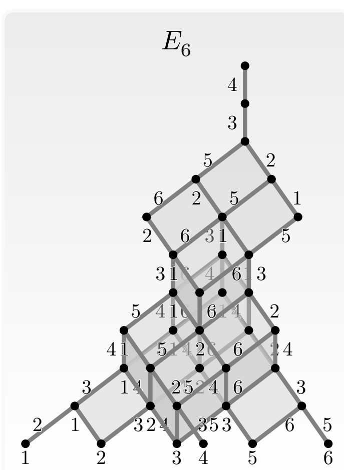

A_1
A_2
A_3
A_4
A_5
A_6

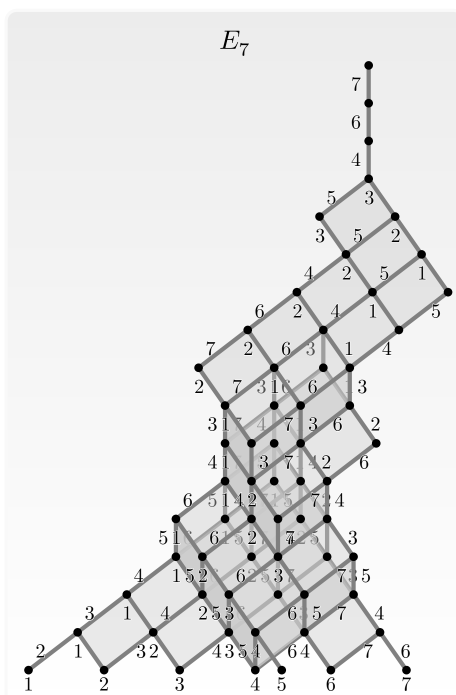

B_3
B_4
B_5

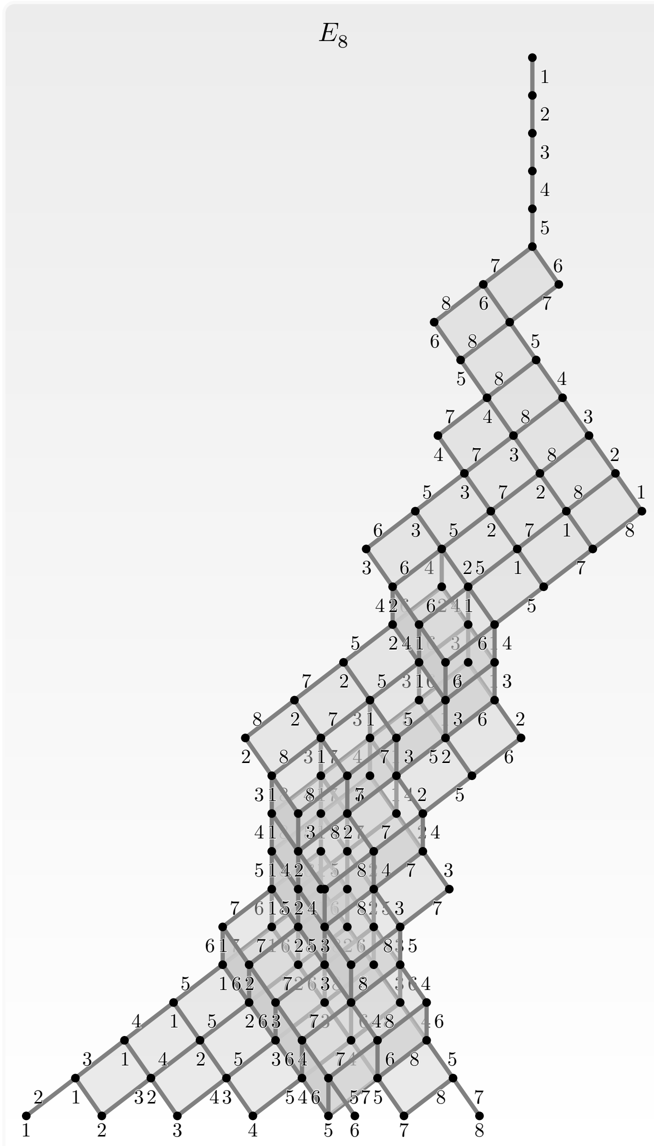

C_2
C_3
C_4

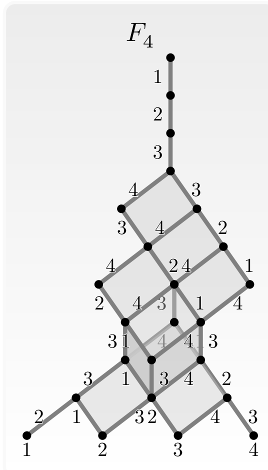

C_5

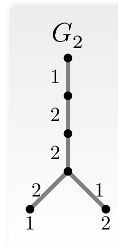

C_6


E_6


E_7


E8





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